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Essays on Repated Moral Hazard

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Abstract

In Chapter 1, I study an infinitely repeated moral hazard problem in which the principal privately observes and publicly reports the agent's output, as in Fuchs (2007). The role of the agent's private strategies, which depend on the history of his private efforts, is examined in providing incentives for the principal to be truthful. I show that in order for his effort history to work as an incentive device, the agent has to use a mixed strategy, since otherwise his efforts are predictable by the principal and thus, in effect, public information. However, hiding the agent's efforts from the principal incurs a non-negligible efficiency loss, which may, or may not be justified by the efficiency gain from the use of the agent's private strategies. Moreover, the agent's optimal strategy is shown to be consistent with empirical studies on how employees respond to subjective performance evaluations.

In Chapter 2, we studies an equilibrium model of the labor market with moral hazard in which jobs are dynamic contracts, job separations are terminations of optimal dynamic contracts, and terminations are used as an incentive device. Transitions from unemployment to new jobs are modeled as a process of matching and bargaining. Non-employed workers make consumption and saving decisions as in a typical growth model, but they must also decide whether or not to participate in the labor market. The equilibrium of the model is characterized. We then calibrate the model to the U.S. labor market to study quantitatively worker turnover, compensation dynamics and distribution. We show that the model can generate equilibrium wage dispersions similar to that in the data. Hornstein, Krusell and Violante (2006) argue that standard search matching models

can generate only a very small differential between the average wage and the lowest wage paid in the labor market.

1 Repeated Moral Hazard with Private Evaluation: Why the Agent's Mixed Strategies Matter

1.1 Introduction

Subjective performance measures are widely used in practice, as observed by Prendergast (1999). For instance, the quality of an analyst's research report is subjectively evaluated by his supervisor. Subjective evaluations are essentially private, therefore non-verifiable by outsiders. Consequently, employers have incentives to underreport performance measures in order to save on wages. In addition, employees may regard performance evaluations as either being fair or not.

Adams (1963) was among the first to suggest that when an employer's evaluation of an employee's performance does not match the employee's own evaluation, the employee regards it as being unfair and exerts less effort in the future. The link between perceived fairness and subsequent performance is confirmed empirically by Ball, Trevino and Sims (1994). Moreover, Greenberg (1986) shows that an employee is more likely to regard a positive evaluation as being fair than a negative one.

In short, an employee's performance depends on the perceived fairness of his prior performance evaluation by an employer, which in turn depends on how he evaluates his own prior performance. Because the employee's self-evaluation is essentially private, it is natural to model this via private strategies.

However, in the standard repeated principal-agent problem, the agent's output is publicly observable to both parties. As a result, no efficiency is lost if we restrict the agent to public strategies, which only depend on the public history of outputs. This is true no matter whether the agent's output is verifiable by outsiders, as in Spear and Srivastava (1987), or not, as in Baker, Gibbons and Murphy (1994)[1].

When the agent's output is not publicly observable, each party may have private information about it. By assuming that each party receives a private signal about the agent's output, Macleod (2003) studies optimal static contracting with variable degrees of signal correlation. He shows that as long as the agent's signal is informative about the principal's, the agent can use it to provide incentives for the principal to be truthful. Specifically, the principal is punished for evaluating the agent's performance as being unsatisfactory, if the agent evaluates his own performance otherwise.

I extend Macleod (2003) to the dynamic environment in which the principal privately observes and publicly reports the agent's output, as in Levin (2003) and Fuchs (2007). Both parties are risk neutral such that total payoff is used to measure efficiency. This is a repeated game with private monitoring which lacks a recursive

[1]See Levin (2003) for more details.

representation, as in Abreu, Pearce and Stacchetti (1991) for repeated games with public monitoring[2]. The agent's private information is the history of his efforts and the principal's private information is the history of true outputs. The agent's self-evaluation is defined directly as his effort, instead of the private signal received as in Macleod (2003).

I first show that if the agent uses a pure strategy as in Fuchs (2007), his efforts are predictable by the principal and thus, in effect, public information. Therefore, again, no efficiency is lost if we restrict the agent to public strategies, when it comes to weak perfect Bayesian equilibria, as defined in Mas-colell, Whinston and Green (1995). To see this, first notice that the principal perfectly predicts what the agent's effort is in the first period, as specified by the agent's pure strategy. And, the agent's effort in the second period is predicted as a deterministic function of the reported output and the agent's effort in the first period, and so on. Hence, the principal's belief about the agent's effort history is degenerate, and independent of the history of true outputs. Moreover, because the principal's belief is independent of the history of true outputs, she reports truthfully only if she is indifferent between reporting the low and high outputs[3].

When it comes to sequential equilibria as considered in this paper, the agent's strategies are most likely to be private. The reason is that different private effort histories of the agent generate different beliefs, therefore different optimal continuation strategies. However, given that the agent uses a pure strategy, the set of allocations attainable by sequential equilibria is shown to be the same as the set of allocations attainable by weak perfect Bayesian equilibria in which the agent is restricted to public strategies.

To summarize, the agent's use of mixed strategies is necessary for his prior efforts to be private, therefore potentially effective in providing incentives for the principal. Moreover, I show that by using mixed strategies, the agent is able to provide stronger incentives in the sense that the principal strictly prefers reporting truthfully.

I consider the optimal perfect public equilibrium as the benchmark case, which have been partially characterized by Levin (2003) and Fuchs (2007) under the constraints such as, the principal reports truthfully, and the agent is indifferent between shirking and exerting effort[4]. I solve for the optimal perfect public equilibrium ex-

[2]Even for repeated games with public monitoring, the recursive representation does not hold if players use mixed strategies. Also, see Kandori (2002) for an excellent introduction on repeated games with private monitoring.

[3]Because the principal's belief about the agent's effort history is independent of the history of true outputs, truth-reporting basically requires her to take different actions given the same belief. This is possible only if each action taken is optimal. Specifically, both reporting the low and high outputs have to be optimal for the principal.

[4]The first constraint is imposed by both Levin (2003) and Fuchs (2007). In addition, the second

plicitly without such constraints. Specifically, I generalize the method of Radner, Myerson and Maskin (1986) to derive an upper bound on the maximum total payoff attainable by perfect public equilibria. Then, the optimal perfect public equilibrium attaining the upper bound is constructed, which consists of a static contract, a pure strategy for the agent, and a non-truth-reporting strategy for the principal.

The optimal contract is static instead of dynamic, which generalizes the static contract result of Levin (2003) without imposing the constraint that the principal reports truthfully. Moreover, the agent's optimal public strategy is pure instead of mixed, therefore his efforts are predictable by the principal as argued above. Furthermore, the principal does not reveal the low output with probability one. As suggested by Prendergast (1999), this phenomenon is well documented in empirical studies as leniency bias, which implies that an supervisor tends to overstate an subordinate's performance. This reflects the fact that the low output occurs with positive probability, even when the agent exerts effort. In other words, the low output is not a perfect indicator for shirking.

Moreover, Abreu, Milgrom and Pearce (1991) show that it is optimal for the principal to observe the agent's outputs for more than one period in order to accumulate more information, before rewarding or punishing the agent. Consequently, when the principal delays reporting[5], it becomes difficult for the agent to provide incentives for the principal to be truthful.

In order to focus on how the agent provides incentives for the principal, I also solve for the optimal perfect public equilibrium, in which the principal reports the agent's output truthfully in each period. It is shown that the agent's optimal public strategy depends only on the reported output in the prior period, rather than the complete history of reported outputs. Specifically, the agent exerts effort with probability one if the reported output in the prior period is high, and with a probability less than one otherwise. Moreover, it is shown that no efficiency is lost by requiring the principal to report the agent's output truthfully, as long as the discount factor is larger than a critical value. Intuitively, given that the principal has incentives to report the low output in order to save on compensations, the agent punishes the principal for reporting it by exerting less effort in the next period.

With private strategies, the agent is able to further make the distinction between the reported low output in a period when he shirked, and the reported low output in a period when he exerted effort. Apparently, the latter is a better indicator that the principal lied than the former. However, in order to provide incentives more effectively

one is imposed by Fuchs (2007).

[5]Even though the principal has to report in each period, the reporting delay can be achieved by reporting a certain output regardless of the true output. See Section 3.3 for an example.

for the principal by making the distinction as described above, the agent must hide his efforts from the principal by shirking with positive probability. This is the basic trade-off in this paper.

In order to address this trade-off, I consider a class of sequential equilibria in which the agent's strategy depends on the reported output *and* his effort in the prior period, and the principal reports the agent's output truthfully in each period. Consequently, the agent's strategy can be represented as a finite state automaton[6], and so can the principal's. These are correlated sequential equilibria according to Phelan and Skrzypacz (2008). Notice that the optimal perfect public equilibrium I solved previously can be treated as a special case.

I show that there is only one static contract consistent with this class of sequential equilibria[7]. Moreover, given this static contract and the principal's truth-reporting strategy, it can be shown that the agent is always indifferent between shirking and exerting effort. Therefore, in order to construct an equilibrium, I need only make sure that given the agent's strategy, the principal has incentives to be truthful.

I first consider a subclass of sequential equilibria, in which the principal has incentives to report truthfully, regardless of her belief about whether the agent has shirked or exerted effort. These are belief-free equilibria according to Ely, Hörner and Olszewski (2005). Within this subclass, the agent's optimal strategy is shown to be public, which implies that the agent's effort in the prior period cannot work effectively as an incentive device.

Then, given a belief-based equilibrium in which the principal's incentives depend non-trivially on her belief about whether the agent has shirked or exerted effort, it is shown that the agent's effort is always "private", in the sense that the principal is unable to infer it with certainty under any circumstances. In addition, the agent's output always contains the "right" message, in the sense that there exists a fixed threshold such that no matter what happened in the past, the low (high) output makes the principal believe that the probability of the agent having exerted effort is less (greater) than this threshold.

I further show that the agent's strategy in the optimal belief-free (also public as shown above) equilibrium, cannot be approximated by a sequence of his strategies in belief-based equilibria[8]. The reason is that in the optimal belief-free equilibrium, the principal is indifferent between reporting the low and high outputs, instead of preferring strictly being truthful. Therefore, no matter how small the agent's deviation is from his strategy in the optimal belief-free equilibrium, the principal could lose

[6]Both the set of outputs and the set of efforts are finite in this paper.
[7]In the sense that there exists a sequentital equilibrium of this class with respect to the contract.
[8]The agent's strategy is a point in the space $[0,1]^4$ given two possible outputs and two possible effort levels.

incentives to be truthful[9].

Furthermore, I show that there exists $\varepsilon > 0$ such that in any belief-based equilibrium, the principal cannot expect the agent to exert effort with a probability greater than $1 - \varepsilon$ under any circumstances. On the contrary, in the optimal belief-free equilibrium, upon reporting the high output in the prior period, the principal expects the agent to exert effort with probability one. This is a non-negligible efficiency loss associated with the need of hiding the agent's efforts from the principal. Numerical analysis shows that this efficiency loss may, or may not be justified by the efficiency gain from using the agent's effort in the prior period as an incentive device.

This paper is also related to the literature on the efficiency implications of private strategies in repeated games with public monitoring[10]. I consider belief-based equilibria in an infinitely repeated game, instead of belief-based equilibria in finitely repeated games as in Mailath, Matthews and Sekiguchi (2002), or belief-free equilibria in infinitely repeated games as in Kandori and Obara (2006).

Moreover, the agent's optimal strategy is shown to be consistent with empirical studies, e.g. Greenberg (1986), on how employees respond to subjective performance evaluations. Specifically, conditional on the reported output being low in the prior period, the agent would exert less effort if he exerted effort, other than shirked, in the prior period.

The rest of the paper is organized as follows: Section 1.2 sets up the model. Section 1.3.1 solves for the optimal perfect public equilibria with and without imposing the constraint that the principal reports truthfully. Section 1.3.2 characterizes correlated sequential equilibria. In addition, an extension is considered in Section 1.3.3. Finally, Section 1.4 concludes.

1.2 Setup

Time is indexed by $t = 0, 1, \cdots$. There are an agent and a principal who are both risk neutral. From time to time, I refer to the agent as he and the principal as she. The principal has access to a project demanding the agent's effort as input. Given the agent's effort in period t denoted by $e_t \in E := \{0, 1\}$, his output in period t denoted by $\theta_t \in \Theta := \{\theta_L, \theta_H\}$ with $\theta_L < \theta_H$ is drawn from a time-invariant distribution

$$\pi(\theta_t = \theta_H \mid e_t) = \begin{cases} q\,, \text{ if } \ e_t = 0 \\ p\,, \text{ if } \ e_t = 1 \end{cases} \text{ with } 0 < q < p < 1. \tag{1}$$

[9]See Mailath and Morris (2002) for more information on a related problem.

[10]For the agent's point of view, the monitoring is public because he does not receive any private signal.

Shirking $e_t = 0$ is costless, but exerting effort $e_t = 1$ causes a fixed cost $c > 0$ so that

$$\phi(e_t) = \begin{cases} 0\,, \text{ if } \; e_t = 0 \\ c\,, \text{ if } \; e_t = 1 \end{cases}$$

is the agent's cost function of effort. Assume

$$(1-q)\theta_L + q\theta_H < (1-p)\theta_L + p\theta_H - c \tag{2}$$

which implies that the agent's effort is productive. The agent maximizes

$$(1-\beta)\mathbb{E}_\tau \left[\sum_{t=\tau}^{\infty} \beta^{t-\tau}(w_t - \phi(e_t))\right]$$

and the principal maximizes

$$(1-\beta)\mathbb{E}_\tau \left[\sum_{t=\tau}^{\infty} \beta^{t-\tau}(\theta_t - w_t)\right]$$

where $\beta \in (0,1)$ is the common discount factor, and $w_t \in \mathbb{R}$ is the principal's transfer to the agent in period t[11].

The agent's effort is unobservable to the principal. Due to (1), the principal is unable to infer with certainty whether the agent shirks or exerts effort from his output. In addition, the agent's output is privately observed by the principal.

In each period, the agent either shirks or exerts effort, then the principal reports (truthfully or not) the output. Hence, the agent's action set is E and the principal's action set is Θ^2. The principal's action $(\vartheta, \vartheta') \in \Theta^2$ implies that she reports ϑ (ϑ') when the true output is θ_L (θ_H).

At the beginning of period t, the history of reported outputs $\vartheta^t := (\vartheta_0, \cdots, \vartheta_{t-1}) \in \Theta^t$ is public information. Besides that, the agent has his private history of efforts $e^t := (e_0, \cdots, e_{t-1}) \in E^t$ and the principal has her private history of true outputs $\theta^t := (\theta_0, \cdots, \theta_{t-1}) \in \Theta^t$. Therefore, the agent's complete history is (ϑ^t, e^t) and the principal's complete history is (ϑ^t, θ^t).

A contract is a mapping

$$w : \bigcup_{t=0}^{\infty} \Theta^t \to \mathbb{R}^2$$

as a transfer schedule contingent on public histories. Any contract is enforceable by a third party[12]. Denote the set of possible contracts by W. The agent's behavioral

[11]If $w_t < 0$, the transfer goes from the agent to the principal.
[12]Any contract I construct in this paper is self-enforcing. But it is easier to start with this assumption.

effort strategy is a mapping

$$e : \bigcup_{t=0}^{\infty} \left(\Theta^t \times E^t\right) \to \Delta(E)$$

and the principal's behavioral report strategy is a mapping

$$r : \bigcup_{t=0}^{\infty} \left(\Theta^t \times \Theta^t\right) \to (\Delta\Theta)^2[13].$$

Denote the set of the agent's possible strategies by S^A and the set of the principal's possible strategies by S^P. Hence, a strategy profile is denoted by $s := (e, r) \in S := S^A \times S^P$.

Denote $\omega \mid \vartheta^t \in W$ as the continuation contract following public history ϑ^t, $e \mid (\vartheta^t, e^t) \in S^A$ as the agent's continuation strategy following his history (ϑ^t, e^t) and $r \mid (\vartheta^t, \theta^t) \in S^P$ as the principal's continuation strategy following her history (ϑ^t, θ^t). Denote $w[\vartheta^t](\vartheta_t)$ as the transfer in period t contingent on reported output ϑ_t following public history ϑ^t, $e[\vartheta^t, e^t](e_t)$ as the probability of the agent with history (ϑ^t, e^t) exerting effort e_t[14] in period t, and $r[\vartheta^t, \theta^t; \theta_t](\vartheta_t)$ as the probability of the principal with history (ϑ^t, θ^t) reporting ϑ_t in period t when observing true output θ_t.

Denote $U^A(s, \omega)$ and $U^P(s, \omega)$ as the normalized expected discounted (NED thereafter) payoffs for the agent and the principal respectively given (continuation) strategy profile s and (continuation) contract ω.

Given strategy profile s, denote $\varphi[\vartheta^t, e^t; s](\theta^t) \in [0, 1]$ as the probability assigned by the agent with history (ϑ^t, e^t) to the principal having private history θ^t (or having complete history (ϑ^t, θ^t)) and $\mu[\vartheta^t, \theta^t; s](e^t) \in [0, 1]$ as the probability assigned by the principal with history (ϑ^t, θ^t) to the agent having private history e^t (or having complete history (ϑ^t, e^t)). Apparently,

$$\sum\nolimits_{\theta^t} \varphi[\vartheta^t, e^t; s](\theta^t) = 1 \text{ and } \sum\nolimits_{e^t} \mu[\vartheta^t, \theta^t; s](e^t) = 1.$$

Furthermore, at the beginning of period 0,

$$\varphi[\vartheta^0, e^0; s](\theta^0) = 1 \text{ and } \mu[\vartheta^0, \theta^0; s](e^0) = 1$$

where ϑ^0, e^0 and θ^0 are empty sets[15]. The agent's and principal's beliefs evolve as

[13]Or equivalently, $\Delta(\Theta^2)$.
[14]Exerting effort 0 means shirking.
[15]In Section 3.2, I assume that at the beginning of peroid 0, Nature draws the initial states for the agent and the principal respectively so that each party holds a non-degenerate belief about what the other's initial state is.

follows,

$$\varphi[\vartheta^{t+1}, e^{t+1}; s](\theta^{t+1}) = \frac{\varphi[\vartheta^t, e^t; s](\theta^t) r[\vartheta^t, \theta^t; \theta_t](\vartheta_t) \pi(\theta_t \mid e_t)}{\sum_{\overline{\theta}^{t+1}} \varphi[\vartheta^t, e^t; s](\overline{\theta}^t) r[\vartheta^t, \overline{\theta}^t; \overline{\theta}_t](\vartheta_t) \pi(\overline{\theta}_t \mid e_t)} \tag{3}$$

$$\mu[\vartheta^{t+1}, \theta^{t+1}; s](e^{t+1}) = \frac{\mu[\vartheta^t, \theta^t; s](e^t) e[\vartheta^t, e^t](e_t) \pi(\theta_t \mid e_t)}{\sum_{\overline{e}^{t+1}} \mu[\vartheta^t, \theta^t; s](\overline{e}^t) e[\vartheta^t, \overline{e}^t](\overline{e}_t) \pi(\theta_t \mid \overline{e}_t)} \tag{4}$$

respectively. However, the denominator in (3) may be zero (not necessarily) off the equilibrium path because the full support assumption is not satisfied. For instance, if the principal's strategy is to report low output θ_L regardless of true output θ_t, then reported output $\vartheta_t = \theta_H$ as a public signal occurs with probability zero. If (3) is not well-defined for some $(\vartheta^{t+1}, e^{t+1})$, then I define

$$\varphi[\vartheta^{t+1}, e^{t+1}; s](\theta^{t+1}) = \prod_{\tau=0}^{t} \pi(\theta_\tau \mid e_\tau) \tag{5}$$

instead. In addition, following $(\vartheta^{t+1}, e^{t+1})$, the agent's belief is defined by (5) as well. In words, as long as the agent detects that the principal has deviated from her strategy r, his belief becomes independent of the principal's reports thereafter. Given any strategy profile s, $\mu[\vartheta^{t+1}, \theta^{t+1}; s](e^{t+1})$ is well-defined even off the equilibrium path due to (1), which implies that the agent's potential deviations are undetectable to the principal. It can be shown that $((e, r), (\varphi, \mu))$ is consistent, as defined in Fudenberg and Tirole (1991).

Definition 1.1 *A strategy profile (e, r) is a sequential equilibrium (SE) with respect to w if*

$$e \mid (\vartheta^t, e^t) \in \arg\max_{e' \in S^A} \left\{ \sum\nolimits_{\theta^t} \varphi[\vartheta^t, e^t; s](\theta^t) U^A((e', r \mid (\vartheta^t, \theta^t)), w \mid \vartheta^t) \right\} \;\forall \vartheta^t, e^t, \tag{6}$$

$$r \mid (\vartheta^t, \theta^t) \in \arg\max_{r' \in S^P} \left\{ \sum\nolimits_{e^t} \mu[\vartheta^t, \theta^t; s](e^t) U^P((e \mid (\vartheta^t, e^t), r'), w \mid \vartheta^t) \right\} \;\forall \vartheta^t, \theta^t, \tag{7}$$

while (φ, μ) is defined by (3)-(5).

1.3 Analysis

The agent's effort affects the likelihood of the principal observing low (high) output θ_L (θ_H) so that it can be used to infer whether the principal is being truthful or not. For instance, when the principal reports low output θ_L, the agent should assign a higher probability to the principal being truthful if he shirked, than if he exerted effort due to $1-q > 1-p$. Therefore, the agent's private history of efforts e^t is expected to play an active role in providing incentives for the principal to be truthful. However, when the agent uses a pure strategy as in Fuchs (2007), a payoff-equivalence exists between a sequential equilibrium and a weak perfect Bayesian equilibrium (WPBE)[16] in which the agent is restricted to public strategies, as shown in Proposition 1.1. Thus, the agent's private history is not necessarily effective as an incentive device. The reason is that when the agent uses a pure strategy, his private history is not really "private" in the sense that the principal's belief about it is degenerate.

Proposition 1.1 *Suppose that the agent uses a pure strategy. Given contract w, denote $V_{SE}(w)$ as the set of SE payoff pairs, $V^1_{WPBE}(w)$ as the set of WPBE payoff pairs, and $V^2_{WPBE}(w)$ as the set of payoff pairs attainable by WPBE in which the agent is restricted to public strategies. Then*

$$V_{SE}(w) = V^1_{WPBE}(w) = V^2_{WPBE}(w).$$

Proof. See Appendix 1.5.1. ■

I have $V_{SE}(w) \subseteq V^1_{WPBE}(w)$ because SE is a refinement of WPBE and $V_{SE}(w) \supseteq V^1_{WPBE}(w)$ if the full support assumption is satisfied, which implies that each party's potential deviations are undetectable to the other. However, that is not the case in this paper so that I have to deal with beliefs off the equilibrium path with caution.

Notice that θ^t has the full support on Θ^t regardless of the agent's strategy due to $q, p \in (0,1)$. So whether the principal's history (ϑ^t, θ^t) is on the equilibrium path depends exclusively on her own strategy in the first t periods. Given public history ϑ^t, if (ϑ^t, θ^t) $\forall \theta^t$ is off the equilibrium path, then (ϑ^t, e^t) $\forall e^t$ is off the equilibrium path too because otherwise there must exist θ^t such that (ϑ^t, θ^t) is on the equilibrium path. In this case, I can redefine the agent's continuation strategy as shirking and the principal's continuation strategy as minimizing the NED transfer with respect to $\omega \mid \vartheta^t$[17]. This is a SE with respect to $\omega \mid \vartheta^t$ which generates the minmax NED payoffs for the agent and the principal respectively. Therefore, neither party has incentives to deviate from his/her original strategy to put some weights on his/her minmax payoff with respect to $\omega \mid \vartheta^t$. Actually, the agent is unable to bring ϑ^t back on the equilibrium path because the principal's strategy in the first t periods is the same. If (ϑ^t, θ^t) is on the equilibrium path for some θ^t, the common argument applies because beliefs are well defined by (3) and (4). So each party is able to change his/her strategy off the equilibrium path without interfering the other's incentives on/off the

[16]See Mas-colell, Whinston and Green (1995) for the formal definition of weak perfect Bayesian equilibrium.

[17]Or simply, reporting low output θ_L regardless of real output θ_t if $\omega \mid \vartheta^t$ is static with $w_L \leq w_H$. See Definition 2 for the formal definition of static contract.

equilibrium path. Recursively applying the procedure described from period 1 gives a payoff-equivalent SE.

Regarding the second relation, it suffices to show $V^1_{WPBE}(w) \subseteq V^2_{WPBE}(w)$. Notice that the principal with history (ϑ^t, θ^t) on the equilibrium path[18] assigns probability one to the agent having private history e^t defined recursively as $e_0 = e(\emptyset)$ and

$$e_{\tau+1} = e((\vartheta_0, \cdots, \vartheta_\tau), (e_0, \cdots, e_\tau)) \; \forall \tau = 0, \cdots, t-2 \tag{8}$$

where e is a deterministic function because the agent uses a pure strategy. Then $(\vartheta^t, \bar{e}^t) \; \forall \bar{e}^t \neq e^t$ is off the equilibrium path. So we can replace $e \mid (\vartheta^t, \bar{e}^t) \; \forall \bar{e}^t \neq e^t$ with $e \mid (\vartheta^t, e^t)$ because WPBE does not impose any restrictions on beliefs off the equilibrium path[19]. Recursively applying the procedure described from period 1 gives a payoff-equivalent WPBE in which the agent is restricted to public strategies.

Therefore, even the agent's strategy in a SE is most likely to depend on his private history e^t because different private histories give him different beliefs, therefore different optimal continuation strategies. Proposition 1.1 suggests that the agent's private history of efforts is not effective in providing incentives for the principal to be truthful. The reason is that when the agent uses a pure strategy, his private history is not really "private" in the sense that the principal has a degenerate belief about it as shown above. So the principal only cares about the agent's continuation strategy following the single private history she believes of the agent having. Furthermore, because the principal's belief is independent of the history of true outputs in (8), she reports truthfully only if she is indeed indifferent between reporting low output θ_L and reporting high output θ_H as in Fuchs (2007). Moreover, I show in Section 3.2 that the agent's use of mixed strategies is able to make the principal strictly prefer reporting truthfully.

Proposition 1.2 *Suppose that (e, r) is a SE with respect to w. Then, there exists a payoff-equivalent SE (e', r') with respect to w' such that $w'[\vartheta^t](\vartheta_t = \theta_L) \leq w'[\vartheta^t](\vartheta_t = \theta_H) \; \forall \vartheta^t$.*

Proof. See Appendix 1.5.2. ■

By treating θ_L and θ_H as two public signals the principal has access to, Proposition 1.2 shows that without loss of generality, I can define either one as the one associated with the bonus throughout the contract. Again, I have to be careful about the agent's

[18]If $(\vartheta^t, \theta^t) \; \forall \theta^t$ is off the equilibrium path, the same procedure as described above applies. Notice that it is a public strategy for the agent to shirk under any circumstances.

[19]I can just assume that the agent with history $(\vartheta^t, \bar{e}^t) \; \forall \bar{e}^t \neq e^t$ has the same belief on the principal's private history, as the agent with history (ϑ^t, e^t).

beliefs off the equilibrium path defined by (5) instead of (3). Fortunately, (5) is robust to the nominal change because it is independent of ϑ^{t+1}.

In Section 3.1, I solve for the optimal perfect public equilibria with and without imposing the constraint that the principal reports truthfully, in which the agent's private history of efforts is not used as an incentive device.

1.3.1 Perfect Public Equilibria

Following Fudenberg, Levine and Maskin (1994), I say the agent's strategy e is public if it depends on public history ϑ^t but not on his private history e^t, and the principal's strategy r is public if it depends on public history ϑ^t but not on her private history θ^t. A perfect public equilibrium (PPE) is a SE in which both e and r are public.

Definition 1.2 *A contract w is static if $w[\vartheta^t](\vartheta_t)$ is independent of ϑ^t.*

Given static contract $w = (w_L, w_H) \in \mathbb{R}^2$, denote $V(w)$ as the set of feasible payoff pairs. Without loss of generality, assume $w_L \leq w_H$ according to Proposition 1.2. The agent's minmax payoff is defined by

$$\min_{r\in[0,1]^2}\max_{e\in\{0,1\}} \begin{bmatrix} e \\ 1-e \end{bmatrix}^T \begin{bmatrix} p & 1-p \\ q & 1-q \end{bmatrix} \begin{bmatrix} r_H & 1-r_H \\ r_L & 1-r_L \end{bmatrix} \begin{bmatrix} w_H \\ w_L \end{bmatrix} - ec$$

and the principal's minmax payoff is defined by

$$\min_{e\in[0,1]}\max_{r\in\{0,1\}^2} \begin{bmatrix} e \\ 1-e \end{bmatrix}^T \begin{bmatrix} p & 1-p \\ q & 1-q \end{bmatrix} \left(\begin{bmatrix} \theta_H \\ \theta_L \end{bmatrix} + \begin{bmatrix} r_H & 1-r_H \\ r_L & 1-r_L \end{bmatrix} \begin{bmatrix} -w_H \\ -w_L \end{bmatrix} \right)$$

where $e \in [0,1]$ is the probability of the agent exerting effort, and $r = (r_L, r_H) \in [0,1]^2$ are the probabilities of the principal reporting high output θ_H when true output θ_t is low and high respectively. If $r = (0,1)$, then the principal reports truthfully. Therefore, the minmax payoffs for the agent and the principal are w_L and $\underline{\theta} - w_L$ respectively. Denote

$$V^*(w) := \{(U^A, U^P) \in V(w) \mid U^A \geq w_L \text{ and } U^P \geq \underline{\theta} - w_L\}$$

as the set of individually rational payoff pairs. Moreover, $U^A + U^P$ is used to measure efficiency because both parties are risk neutral.

Define a mapping $\Pi : 2^{\mathbb{R}^2} \to 2^{\mathbb{R}^2}$ as

$$\Pi(X) = \left\{ (U^A, U^P) \left| \begin{array}{c} \exists U_L, U_H \in co(X)[20],\ e \in [0,1] \text{ and } r \in [0,1]^2 \\ \text{such that} \\ U^A = U^A((e,r), w; U_L, U_H) \\ U^P = U^P((e,r), w; U_L, U_H) \\ e \in \underset{e' \in [0,1]}{\arg\max} U^A((e',r), w; U_L, U_H) \\ r \in \underset{r' \in [0,1]^2}{\arg\max} U^P((e,r'), w; U_L, U_H) \end{array} \right. \right\} \quad (9)$$

where

$$U^A((e,r), w; U_L, U_H) \\ = \begin{bmatrix} e \\ 1-e \end{bmatrix}^{\intercal} \begin{bmatrix} p & 1-p \\ q & 1-q \end{bmatrix} \begin{bmatrix} r_H & 1-r_H \\ r_L & 1-r_L \end{bmatrix} \begin{bmatrix} (1-\beta)w_H + \beta U_H^A \\ (1-\beta)w_L + \beta U_L^A \end{bmatrix} - e(1-\beta)c$$

$$U^P((e,r), w; U_L, U_H) \\ = \begin{bmatrix} e \\ 1-e \end{bmatrix}^{\intercal} \begin{bmatrix} p & 1-p \\ q & 1-q \end{bmatrix} \left(\begin{bmatrix} (1-\beta)\theta_H \\ (1-\beta)\theta_L \end{bmatrix} + \begin{bmatrix} r_H & 1-r_H \\ r_L & 1-r_L \end{bmatrix} \begin{bmatrix} -(1-\beta)w_H + \beta U_H^P \\ -(1-\beta)w_L + \beta U_L^P \end{bmatrix} \right)$$

so that $\Pi^\infty(V(w))$ is the set of PPE (with respect to w) payoff pairs according to Abreu, Pearce and Stacchetti (1991).

I derive an upper bound on the total payoff attainable by PPE in Proposition 1.3, then prove that it is the least upper bound by constructing the optimal PPE which attains it in Theorem 1.1.

Condition 1 $(p-q)^2(\theta_H - \theta_L) \geq (1-q)c$[21].

Proposition 1.3 *Given a static contract w, denote*

$$\overline{U}(w) = \max U^A + U^P \ \ s.t.\ (U^A, U^P) \in \Gamma^\infty(V(w))$$

as the maximum total payoff attainable by PPE with respect to static contract w. Then,

$$\overline{U}(w) \begin{cases} = \underline{\theta} & ,\ for\ \ \beta \in \big(0, \underline{\beta}(w)\big) \\ \leq \overline{\theta} - \frac{1-q}{p-q}c & ,\ for\ \ \beta \in \big[\underline{\beta}(w), 1\big) \end{cases}$$

[20]$co(X)$ is the convex hull of X which implies that there exists a public randomization device. However, it is not needed to construct the optimal PPE in Theorem 1.

[21]In Proposition 3 and Proposition 4, I prove that if this condition is violated, there does not exist any PPE with respect to any contract in which the agent ever exerts effort.

where

$$\underline{\beta}(w) = \frac{\max\{c, (p-q)(w_H - w_L)\}}{\max\{c, (p-q)(w_H - w_L)\} + (p-q)^2(\theta_H - \theta_L) - (1-q)c} \in [0,1].$$

Proof. See Appendix 1.5.3. ■

Given static contract w, the blue parallelogram in Figure 1.1a represents V the set of feasible payoff pairs. And the shadowed area represents $\Pi^{\infty}(V)$ the set of PPE (with respect to w) payoff pairs including $(w_L, \underline{\theta} - w_L)$ because the agent shirks and the principal reports low output θ_L regardless of true output θ_t is a Nash equilibrium in the stage game. I generalize the method of Radner, Myerson and Maskin (1986) (RMM thereafter) by considering the fact that any continuation payoff pair must Pareto dominate the minmax payoff pair $(w_L, \underline{\theta} - w_L)$. Hence, the upper bound derived is not completely independent of the discount factor as in RMM. That is because the minmax payoff pair imposes a bound on the feasible punishments, which is going to bind if the discount factor becomes too small, or equivalently, the punishments needed become too large.

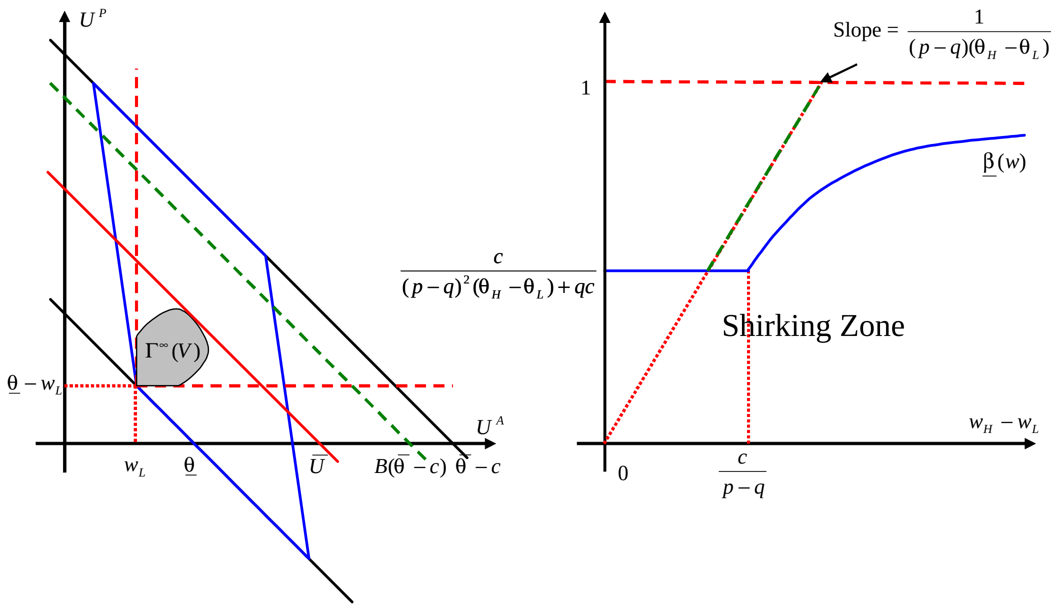

Figure 1.1a and 1.1b

In Figure 1.1b, the shirking zone is located between the blue curve of $\underline{\beta}(\cdot)$ and the horizontal axis where there does not exist any PPE with respect to any static contract in which the agent ever exerts effort. That is the case if either the discount factor becomes too small, specifically less than $\frac{c}{(p-q)^2(\theta_H - \theta_L) + qc}$, or $w_H - w_L$ becomes too large. As a comparison, recall in the case of the agent's output being publicly observable, the agent shirks if $w_H - w_L$ is less than $\frac{c}{p-q}$ and exerts effort otherwise[22]. Therefore, larger $w_H - w_L$, higher incentives for the agent to exert effort. But in this model, the agent's incentives to exert effort depend on not only how large $w_H - w_L$ is, but also on how willing the principal is to be truthful. Unfortunately, the principal's willingness is negatively related to how large she can gain from lying which is also $w_H - w_L$.

I construct the optimal PPE which attains the upper bound $\overline{\theta} - \frac{1-q}{p-q}c$ for the largest set of discount factor values $\left[\frac{c}{(p-q)^2(\theta_H-\theta_L)+qc}, 1\right)$ in Theorem 1.1. The optimal PPE features a pure strategy for the agent, and a non-truth-reporting strategy for the principal.

Theorem 1.1 *(The optimal static contract) Define static contract w^* as $w_L^* = 0$ and $w_H^* = \beta(p-q)(\theta_H - \theta_L)$. Define an automaton with two states: a cooperation state C in which $e = 1$, $r_L = 1 - \frac{1-\beta}{\beta}\frac{c}{(p-q)^2(\theta_H-\theta_L)-(1-q)c}$, $r_H = 1$, and a defection state D in which $e = 0$, $r_L = r_H = 0$. Let C be the initial state. The transition from C to D occurs if and only if low output θ_L is reported. But C is inaccessible to D which implies that as far as D is reached, it prevails forever. Then, this is a PPE with respect to w^* which attains the total NED payoff $\overline{\theta} - \frac{1-q}{p-q}c$[23] for $\beta \in \left[\frac{c}{(p-q)^2(\theta_H-\theta_L)+qc}, 1\right)$.*

Proof. See Appendix 1.5.4. ■

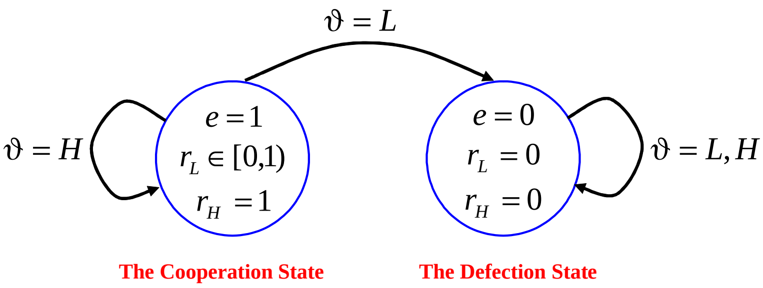

Figure 1.2

The defection state features the Nash equilibrium in the stage game. At the cooperation state, the principal is indifferent between reporting low output θ_L and reporting high output θ_H. Hence, r_L is chosen to be small enough so that the agent has incentives to exert effort, but not too small so that the unnecessary punishments

[22]The agent is indifferent between shirking and exerting effort if $w_H - w_L = \frac{c}{p-q}$.
[23]As p goes to 1 which implies that shirking is detectable when real output θ_t is low, the first best $\overline{\theta} - c$ is achieved.

are triggered. As a result, the agent is also indifferent between shirking and exerting effort.

The green dashed interval with slope $\frac{1}{(p-q)(\theta_H-\theta_L)}$ in Figure 1.1b represents the optimal static contract characterized by $w_H^* - w_L^*$. It is interesting to observe that when the discount factor is just slightly larger than $\frac{c}{(p-q)^2(\theta_H-\theta_L)+qc}$, the agent exerts effort even when $w_H^* - w_L^*$ is strictly less than $\frac{c}{p-q}$. On the contrary, if the agent's output is publicly observable, the agent never exerts effort if $w_H - w_L < \frac{c}{p-q}$. The difference is that when the agent's output is publicly observable, the agent's expected payoff from shirking is $(1-q)w_L + qw_H$ guaranteed. But in this model, the principal can punish the agent even further by reporting low output θ_L regardless of true output θ_t so that the agent's expected payoff from shirking is w_L instead. In some sense, the principal is able to fire the agent[24]. Therefore, as long as the relationship is still valuable to the agent in the sense that $w_H - w_L$ is not too small, or specifically $(1-p)w_L + pw_H > w_L$[25], the agent has incentives to exert effort in order to stay in.

At the cooperation state, $r_L \in [0, 1)$ and $r_H = 1$ imply that high output $\theta_t = \theta_H$ is always rewarded, but low output $\theta_t = \theta_L$ is not always punished. That is because the role of reported low output $\vartheta_t = \theta_L$ is two fold: it differentiates the agent's output ex post in order to provide ex ante incentives. But it also serves as the public signal upon which the agent and the principal can coordinate their future actions, or specifically, their future mutual punishments as described at the defection state. As far as the efficiency is concerned, it is not optimal to trigger the mutual punishments more frequently than necessary in terms of providing incentives for the agent to exert effort. Therefore, it is not surprising to observe that larger the discount factor, larger the bias measured by r_L because when the agent values his future payoff more, a smaller probability of triggering the punishments is enough to provide incentives for the agent. As suggested by Prendergast (1999), this phenomenon is well documented in empirical studies as leniency bias, which implies that an supervisor tends to overstate an subordinate's performance.

This reflects the fact that the principal is unable to observe the agent's effort directly. Otherwise, there is no efficiency loss of threatening to punish the agent as severely as possible because the punishments will not be mistakenly triggered. Abreu, Milgrom and Pearce (1991) (AMP thereafter) further show that it is optimal for the principal to observe the agent's outputs for more than one period in order to accumulate more information before rewarding or punishing the agent. For instance, in the T-period review equilibrium replicated from Fuchs (2007) without referring

[24]This implies allowing the principal to be able to fire the agent explicitly in this model may not be necessary.

[25]This requires $w_H - w_L > \frac{c}{p}$ which is apparently weaker than $w_H - w_L > \frac{c}{p-q}$.

to termination, the principal reports high output θ_H regardless of true output θ_t for the first $T-1$ periods[26]. Hence, the agent receives a signal informative about the principal's action for every T periods. And the principal has incentives to keep T as large as possible (T can go to infinity as the discount factor goes to one). This illustrates an informational disadvantage for the agent in the sense that the principal observes true output θ_t informative about the agent's effort every period, but is able to keep the agent from receiving information about her action frequently. In order to focus on how the agent provides incentives for the principal, I impose the truth-reporting constraint upon the principal. So the agent is the only party keeping private information on the equilibrium path. This is also consistent with the common practice of employee participation in the performance evaluation process by keeping employees informed as documented in Cawley, Keeping and Levy (1998).

Also, notice that the agent uses a pure strategy in Theorem 1.1 so that his private history of efforts is actually public, therefore not effective in providing incentives for the principal to be truthful according to Proposition 1.1. That suggests that making it private therefore potentially effective may impose restrictions on the agent's strategies. Unlike Macleod (2003) in which the agent's self-evaluation, defined as a private signal received, is private by assumption, keeping the agent's self-evaluation defined as his effort private may come with a efficiency loss.

Consider an example based on Theorem 1.1 to illustrate how the agent's use of mixed strategies is able to generate more information for the agent with a efficiency loss. Notice the agent at state C observes reported low (high) output $\vartheta_t = \theta_L$ (θ_H) with probability $(1-p)(1-r_L)$ $(p+(1-p)r_L)$. This can be the result of the principal's strategy being $(r_L, 1)$ as defined in Theorem 1.1 or $(p+(1-p)r_L, p+(1-p)r_L)$ in which the principal reports high output θ_H with probability $p+(1-p)r_L$ regardless of true output θ_t. If it is the latter, the agent's optimal strategy is to shirk instead. Therefore, the agent has incentives to find out by shirking randomly. If shirking seems not to affect the probability of the agent receiving reported high output $\vartheta_t = \theta_H$, it is more likely that the principal's strategy is the latter so that the agent should shirk.

Proposition 1.4 *Allowing dynamic contracting does not increase the maximum total NED payoff attainable by PPE.*

Proof. See Appendix 1.5.5. ∎

Proposition 1.4 generalizes the static contract result of Levin (2003) without imposing the constraint that the principal reports truthfully[27]. It turns out to be easier

[26]See Section 3.3 for details.

[27]But the role of dynamic contracting is unclear when it comes to SE with the agent's mixed strategies. See Fuchs (2007) for a static contract result with the agent's pure strategies.

to derive the set of PPE payoff pairs with respect to some dynamic contract directly, instead of the set of PPE payoff pairs with respect to a certain dynamic contract. The way to do that is to imagine that there exists a third player called contract generator, for instance, besides the agent and the principal. The contract generator's action set is $\mathbb{R}^2$. And its strategy is a mapping

$$w : \bigcup_{t=0}^{\infty} \Theta^t \to \mathbb{R}^2.$$

As a result, the set of PPE payoff pairs is characterized as an area between two parallel lines because if (U^A, U^P) is in it, then any $(\widetilde{U}^A, \widetilde{U}^P)$ with $\widetilde{U}^A + \widetilde{U}^P = U^A + U^P$ is in it too. All we have to do is to add a constant transfer without interfering incentives for the agent and the principal.

Definition 1.3 *The principal's report strategy features truth-reporting if the principal reports truthfully as long as no false reports have been made previously.*

It is not necessary to be specific about the principal's strategy off the equilibrium path, because given the principal's strategy on the equilibrium path as defined above, the agent never assigns positive probability to the principal being off the equilibrium path due to $p, q \in (0, 1)$.

Lemma 1.1 *Suppose that (e, r) is a SE with respect to static contract $w = (0, w_H)$. If the principal's report strategy features truth-reporting, then $w_H = 0$ or $\frac{c}{p-q}$.*

Proof. See Appendix 1.5.6. ■

The intuition is simple. If the principal's strategy features truth-reporting, then the agent expects to observe any public history with positive probability. So no matter what the agent's history is, he believes with certainty that the principal is on the equilibrium path, therefore is going to be truthful in the future. Given that, if w_H is strictly less (greater) than $\frac{c}{p-q}$, then the agent has a strictly dominant strategy of shirking (exerting effort). In either case, the principal does not have incentives to report high output θ_H as long as it is costly with $w_H > 0$, because the agent's continuation strategy is independent of her report. For notational simplification, let $w^{**} = \left(0, \frac{c}{p-q}\right)$ which is the only static contract I have to consider, when it comes to SE in which the principal reports truthfully.

Because the efficiency is exclusively determined by the agent's effort, it is equivalent to focus on the NED probability of the agent exerting effort defined as

$$\mathbf{e} = (1-\beta)\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t e_t \middle| s \right] \in [0, 1] \tag{10}$$

instead of the total NED payoff. For instance, the total NED payoff $\overline{\theta}-\frac{1-q}{p-q}c$ attainable by the optimal PPE is equivalent to the NED probability of the agent exerting effort $\frac{(p-q)^2(\theta_H-\theta_L)-(1-q)c}{(p-q)[(p-q)(\theta_H-\theta_L)-c]}$. Furthermore, define

$$\mathbb{C} \equiv \left\{ \mathbf{e} \in [0,1] \;\middle|\; \begin{array}{c} \exists \text{a PPE } (e,r) \text{ with respect to } w^{**} \\ \text{such that} \\ (10) \text{ and } r \text{ features truth-reporting} \end{array} \right\} \tag{11}$$

as the set of NED probabilities of the agent exerting effort in PPE with respect to w^{**} in which the principal's strategy features truth-reporting.

Theorem 1.2 *$\mathbb{C}$ is empty for $\beta \in \left(0, \frac{c}{(p-q)^2(\theta_H-\theta_L)}\right)$ [28]. Otherwise,*

$$\mathbb{C} = \left[\frac{qc}{(p-q)[(p-q)(\theta_H-\theta_L)-c]}, \frac{(p-q)^2(\theta_H-\theta_L)-(1-q)c}{(p-q)[(p-q)(\theta_H-\theta_L)-c]}\right].$$

Furthermore, there exist e_L and e_H with $e_L < e_H$ such that

$$e(\vartheta^{t+1}) = \begin{cases} e_L \text{ , if } \vartheta_t = \theta_L \\ e_H \text{ , if } \vartheta_t = \theta_H \end{cases} \quad \forall \vartheta^{t+1}.$$

Proof. See Appendix 1.5.7. ■

In the optimal perfect public equilibrium, I have

$$e_L = 1 - \frac{1}{\beta}\frac{c}{(p-q)^2(\theta_H-\theta_L)} \text{ and } e_H = 1.$$

After observing true output θ_t, the principal with history $(\vartheta^t, \theta^{t+1})$ has to decide whether it is optimal to choose the NED payoff following $((\vartheta^t, \vartheta_t = \theta_H), \theta^{t+1})$ over the one following $((\vartheta^t, \vartheta_t = \theta_L), \theta^{t+1})$ by reporting high output θ_H for the cost $w_H^* - w_L^*$. When it comes to PPE in which the agent uses public strategies, the history of true outputs θ^{t+1} can be ignored by the principal. I have a similar situation as in Proposition 1.1 so that the principal has to be indifferent between reporting low output θ_L and reporting high output θ_H. In some sense, the agent is supposed to "reimburse" the principal for the cost $w_H^* - w_L^*$ by promising to exert more effort following reported high output θ_H. Because both the agent and the principal are risk neutral, a first order Markov strategy as described in Theorem 1.2 can do the trick. And, the agent's tit-for-tat strategy as described above is consistent with empirical studies, e.g. Greenberg (1986), which show that an employee is more likely to regard

[28]This requires $(p-q)^2(\theta_H-\theta_L) \geq c$ which is stronger than Condition 1. Otherwise, $\mathbb{C}$ is empty regardless of the discount factor.

a positive evaluation as being fair than a negative one.

As shown in Figure 1.3, requiring the principal to be truthful does not lower the maximum NED probability of the agent exerting effort attainable, but it does increase the threshold on the discount factor value from $\frac{c}{(p-q)^2(\theta_H-\theta_L)+qc}$ to $\frac{c}{(p-q)^2(\theta_H-\theta_L)}$ so that the punishments needed are not too large to be feasible. This can be treated as a direct efficiency loss from requiring the principal to reveal her private information to the agent.

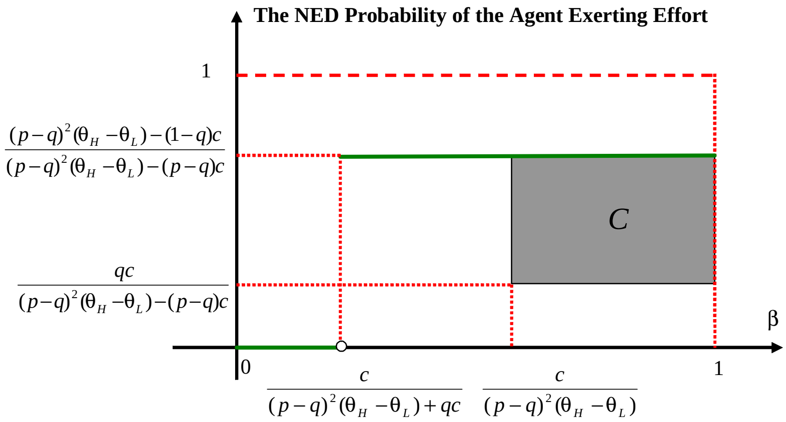

Figure 1.3

In Section 3.2 below, I deviate from the benchmark in Theorem 1.2 by assuming that the agent's strategy is private, which depends on the reported output *and* his effort in the prior period. In this environment, the necessary conditions are characterized under which the agent's effort in the prior period is effective in providing incentives for the principal to be truthful. Then, I identify analytically and measure numerically the potential efficiency loss and gain. Moreover, the agent's optimal strategy is characterized.

1.3.2 Correlated Sequential Equilibria

In this section, I consider private equilibria in which each player's strategy can depend on one's private information, in addition to the public information. Following Phelan and Skrzypacz (2008), I confine my attention to a specific type of private equilibrium, called correlated sequential equilibrium, in which each player's strategy can be represented as a finite state automaton, and the initial states are drawn from a joint distribution at the beginning of the game.

Specifically, I assume that the probability of the agent exerting effort in period $t+1$ is determined by his history in period t, or equivalently, reported output ϑ_t and his effort e_t. Hence, the agent's strategy can be represented as an automaton consisting of a set of states

$$\Gamma = \Theta \times E = \{(L,0),(L,1),(H,0),(H,1)\},$$

an effort function

$$e : \Gamma \to [0,1],$$

where $e(\gamma)$ is the probability of the agent exerting effort at state $\gamma \in \Gamma$, and a transition function $\tau : \Gamma \times \Theta \times E \to \Gamma$ defined as

$$\tau(\gamma, \vartheta, e) = (\vartheta, e),$$

which implies that the agent's state switches to (ϑ, e) in period $t+1$ if and only if his history in period t is (ϑ, e), regardless of his state in period t. For notational simplicity, I refer to $e(\gamma)$ as e_γ thereafter.

In addition, the principal reports the agent's output truthfully in each period. Therefore, the principal's strategy can be represented as an automaton consisting of a set of states

$$\Lambda = \Theta = \{L, H\},$$

a report function $r : \Lambda \times \Theta \to \Theta$ defined as

$$r(\lambda, \theta) = \theta[29],$$

and a transition function $\varsigma : \Lambda \times \Theta \times \Theta \to \Lambda$ defined as

$$\varsigma(\lambda, \vartheta, \theta) = \vartheta,$$

which implies that the principal's state switches to ϑ in period $t+1$ if and only if she reports ϑ in period t, regardless of her state in period t. Notice that the principal's strategy is static in the sense that the report function is independent of her state. Therefore, the certain automaton representation is chosen for technical reasons, which is in no way unique.

At the beginning of period 0, Nature draws the initial states for the agent and principal respectively according to a joint distribution,

	$(L,0)$	$(L,1)$	$(H,0)$	$(H,1)$
L	$\rho_{L,0}$	$\rho_{L,1}$	0	0
H	0	0	$\rho_{H,0}$	$\rho_{H,1}$

[29]This implies that the principal always reports truthfully even off the equilibrium path.

which is common knowledge. Assume $\rho_{L,0} + \rho_{L,1} \neq 0$ and $\rho_{H,0} + \rho_{H,1} \neq 0$. The joint distribution does not have the full support, which is consistent with the fact that, in period $t+1$, the principal at state λ assigns probability zero to the agent being at state $(\lambda', 0)$ and $(\lambda', 1)$ with $\lambda' \neq \lambda$. The reason is that the principal is at state λ in period $t+1$ if and only if she reported λ in period t, which implies that the agent's history in period t (or equivalently, the agent's state in period $t+1$) is either $(\lambda, 0)$ or $(\lambda, 1)$.

The optimal perfect public equilibrium in Theorem 1.2 can be replicated by letting

$$e_{L,0} = e_{L,1} = 1 - \frac{1}{\beta} \frac{c}{(p-q)^2(\theta_H - \theta_L)} \text{ and } e_{H,0} = e_{H,1} = 1$$

$$\rho_{L,0} = \rho_{L,1} = \rho_{H,0} = 0 \text{ and } \rho_{H,1} = 1.$$

According to Lemma 1, without loss of generality, the contract is defined as $w^{**} = \left(0, \frac{c}{p-q}\right)$. Given contract w^{**} and the principal's truth-reporting strategy as defined above, the agent is always indifferent between shirking and exerting effort. As a result, any effort function $e : \Gamma \to [0,1]$ is optimal for the agent. Therefore, I focus on the following question: given the agent's effort function e, does the principal have incentives to be truthful? If the answer is positive, I say that e supports truth-reporting.

I start with defining how the principal's beliefs evolve. Because the agent's continuation strategy is exclusively determined by his state, the principal's relevant belief is about the agent's state $\gamma \in \Gamma$, instead of the agent's private history $e^t \in E^t$. Specifically, the principal's belief at state $\lambda \in \Lambda$ is about whether the agent is at state $(\lambda, 0)$ or $(\lambda, 1)$, or equivalently, whether the agent has exerted effort in previous period (if there is one).

If the principal at state λ assigns probability $x \in [0,1]$ to the agent being at state $(\lambda, 1)$ in period t, she expects the agent to exert effort with probability

$$\overline{e}(x; \lambda) = (1-x)e_{\lambda,0} + xe_{\lambda,1} \in [0,1] \tag{12}$$

before observing true output θ_t. However, after observing true output θ_t to be low and high, the principal assigns probabilities

$$F(x, L; \lambda) = \frac{\overline{e}(x;\lambda)(1-p)}{(1-\overline{e}(x;\lambda))(1-q) + \overline{e}(x;\lambda)(1-p)} \in [0,1] \tag{13}$$

$$F(x, H; \lambda) = \frac{\overline{e}(x;\lambda)p}{(1-\overline{e}(x;\lambda))q + \overline{e}(x;\lambda)p} \in [0,1] \tag{14}$$

to the agent having exerted effort in period t respectively. Notice that (13) and (14) are well-defined due to $q, p \in (0,1)$.

For all $\lambda \in \Lambda$, denote $\mathbb{N}(\lambda) \subseteq [0,1]$ as the set of possible probabilities assigned by the principal at state λ to the agent being at state $(\lambda, 1)$, before observing the true output in current period. Hence, $x \in \mathbb{N}(\lambda)$ if and only if $x = \frac{\rho_{\lambda,1}}{\rho_{\lambda,0}+\rho_{\lambda,1}}$, or there exist initial state $\lambda_0 \in \Lambda$ and history $(\vartheta^{t+1}, \theta^{t+1})$ for the principal such that

$$\vartheta_t = \lambda \text{ and } x = \sum_{e^{t+1} \in E^{t+1} \text{ with } e_t = 1} \mu[\lambda_0; \vartheta^{t+1}, \theta^{t+1}](e^{t+1}),$$

which imply that if the principal starts with λ_0, then following $(\vartheta^{t+1}, \theta^{t+1})$, she is at state λ in period $t+1$, and assigns probability x to the agent having exerted effort in period t, or equivalently, being at state $(\lambda, 1)$ in period $t+1$.

Following Phelan and Skrzypacz (2008), I construct $\mathbb{N}(\lambda)$ by defining $\mathbb{N}_t(\lambda)$ recursively with

$$\mathbb{N}(\lambda) = \bigcup_{t=0}^{\infty} \mathbb{N}_t(\lambda) \ \forall \lambda \in \Lambda \tag{15}$$

as follows: if the principal is at state λ in period 0, then she assigns probability $\frac{\rho_{\lambda,1}}{\rho_{\lambda,0}+\rho_{\lambda,1}}$ to the agent being at state $(\lambda, 1)$ such that

$$\mathbb{N}_0(\lambda) = \left\{ \frac{\rho_{\lambda,1}}{\rho_{\lambda,0}+\rho_{\lambda,1}} \right\}. \tag{16}$$

Furthermore, $\mathbb{N}_{t+1}(\lambda)$ is defined recursively as

$$\begin{aligned} \mathbb{N}_{t+1}(\lambda) &= \Sigma(\mathbb{N}_t(L), \mathbb{N}_t(H); \lambda) \\ &= \left\{ x \in [0,1] \;\middle|\; \begin{array}{c} \exists \lambda' \in \Lambda \text{ and } (\vartheta, \theta) \in \Theta^2 \\ \text{such that} \\ \vartheta = \lambda \text{ and } x = F(x', \theta; \lambda') \text{ for some } x' \in \mathbb{N}_t(\lambda') \end{array} \right\} \end{aligned} \tag{17}$$

where $\Sigma : 2^{[0,1]} \times 2^{[0,1]} \times \Lambda \to 2^{[0,1]}$ is a mapping. The idea is illustrated in Figure 1.4,

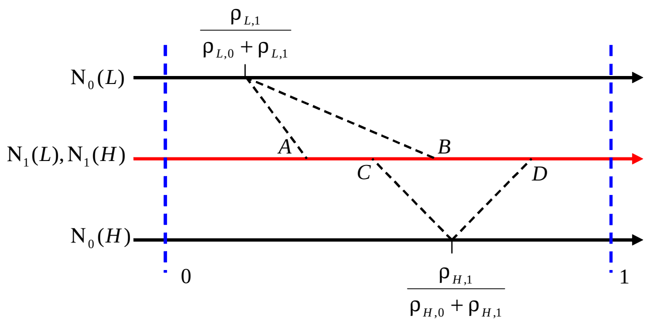

Figure 1.4

in which $\mathbb{N}_1(L) = \mathbb{N}_1(H) = \{A, B, C, D\}$. It is no coincidence to have $\mathbb{N}_1(L) = \mathbb{N}_1(H)$, which will be generalized for $t+1$ by (a) in Proposition 1.8. The idea is that the principal's belief about whether the agent has shirked or exerted effort in period 0 is independent of reported output ϑ_0, because it occurs after the agent makes his effort decision. However, the principal's state in period 1 is exclusively determined by ϑ_0. For instance, if the principal's initial state is L, then she assigns probability $A = F\left(\frac{\rho_{L,1}}{\rho_{L,0}+\rho_{L,1}}, \theta_0 = L; L\right)$ to the agent having exerted effort, after observing true output θ_0 to be low. Then, $A \in \mathbb{N}_1(L)$ if $\vartheta_0 = L$ and $A \in \mathbb{N}_1(H)$ if $\vartheta_0 = H$. The difference is that $A \in \mathbb{N}_1(L)$ is a belief on the equilibrium path, while $A \in \mathbb{N}_1(H)$ is a belief off the equilibrium path because the principal misreported.

Furthermore, for all $\gamma \in \Gamma$, denote $\mathbf{e}_\gamma \in [0, 1]$ as the NED probability of the agent exerting effort, given the agent's continuation strategy at state γ and the principal's truth-reporting strategy. Hence, given that the agent's starting state is γ, $\mathbf{e}_\gamma$ measures how much effort (on average) the principal expects the agent to exert in the future by reporting truthfully. Formally,

$$\mathbf{e} \equiv (1-\beta)e + \beta\mathbb{Z}\mathbf{e} \tag{18}$$

where

$$e \equiv (e_{L,0}\ e_{L,1}\ e_{H,0}\ e_{H,1})^\top$$

$$\mathbf{e} \equiv (\mathbf{e}_{L,0}\ \mathbf{e}_{L,1}\ \mathbf{e}_{H,0}\ \mathbf{e}_{H,1})^\top$$

and

$$\mathbb{Z} \equiv \begin{bmatrix} (1-e_{L,0})(1-q) & e_{L,0}(1-p) & (1-e_{L,0})q & e_{L,0}p \\ (1-e_{L,1})(1-q) & e_{L,1}(1-p) & (1-e_{L,1})q & e_{L,1}p \\ (1-e_{H,0})(1-q) & e_{H,0}(1-p) & (1-e_{H,0})q & e_{H,0}p \\ (1-e_{H,1})(1-q) & e_{H,1}(1-p) & (1-e_{H,1})q & e_{H,1}p \end{bmatrix}$$

is the transition matrix. By algebra, I have

$$\mathbf{e}_\gamma = \frac{e_0}{1-(e_1-e_0)} + \frac{1-\beta}{1-\beta(e_1-e_0)}\left[e_\gamma - \frac{e_0}{1-(e_1-e_0)}\right] \ \forall \gamma \in \Gamma \qquad (19)$$

where $e_0 = (1-q)e_{L,0} + qe_{H,0}$ and $e_1 = (1-p)e_{L,1} + pe_{H,1}$[30]. In addition, $\frac{e_0}{1-(e_1-e_0)}$ is the long term average independent of the agent's starting state, and $\frac{1-\beta}{1-\beta(e_1-e_0)}$ measures how much the agent's starting state matters, which goes to zero as the discount factor goes to one.

Lemma 1.2 *An effort function $e : \Gamma \to [0,1]$ supports truth-reporting if and only if for all $\lambda \in \Lambda$ and $x \in \mathbb{N}(\lambda)$,*

$$(1 - F(x, L; \lambda))(\mathbf{e}_{H,0} - \mathbf{e}_{L,0}) + F(x, L; \lambda)(\mathbf{e}_{H,1} - \mathbf{e}_{L,1}) \leq y \qquad (20)$$

$$(1 - F(x, H; \lambda))(\mathbf{e}_{H,0} - \mathbf{e}_{L,0}) + F(x, H; \lambda)(\mathbf{e}_{H,1} - \mathbf{e}_{L,1}) \geq y \qquad (21)$$

where

$$y \equiv \frac{1-\beta}{\beta}\frac{c}{(p-q)[(p-q)(\theta_H-\theta_L)-c]} > 0[31].$$

Proof. See Appendix 1.5.8. ■

According to Theorem 12.2.2 in Mailath and Samuelson (2006), the one-shot deviation principal applies. So, (20) and (21) are basically no profitable one-shot deviations conditions for the principal, given that the true output is low and high respectively.

Given that the agent has shirked (exerted effort), the principal is able to increase (on average) the probability of the agent exerting effort in the future by $\mathbf{e}_{H,0} - \mathbf{e}_{L,0}$ ($\mathbf{e}_{H,1} - \mathbf{e}_{L,1}$), if she reports the high output for the cost $\frac{c}{p-q}$. However, the principal does not know for sure whether the agent has shirked or exerted effort. Instead, she assigns probability $F(x; L; \lambda)$ ($F(x; H; \lambda)$) to the agent having exerted effort after observing the true output to be low (high). Therefore, the left hand side of (20) ((21)) is how much more effort (on average) the principal expects the agent to exert in the future by reporting the high output in current period, given that the true output

[30]On the equilibrium path, e_0 (e_1) is the expected probability of the agent exerting effort, given that the agent shirks (exerts effort) in previous period.

[31]Apparently, the left hand sides of (20) and (21) are between 0 and 1. Therefore, $y \in [0,1]$ which implies $\beta \geq \frac{c}{(p-q)[(p-q)(\theta_H-\theta_L)-c]+c}$.

is low (high). It is profitable to report the high output if and only if the expected increase, on the probability of the agent exerting effort in the future, is greater than y.

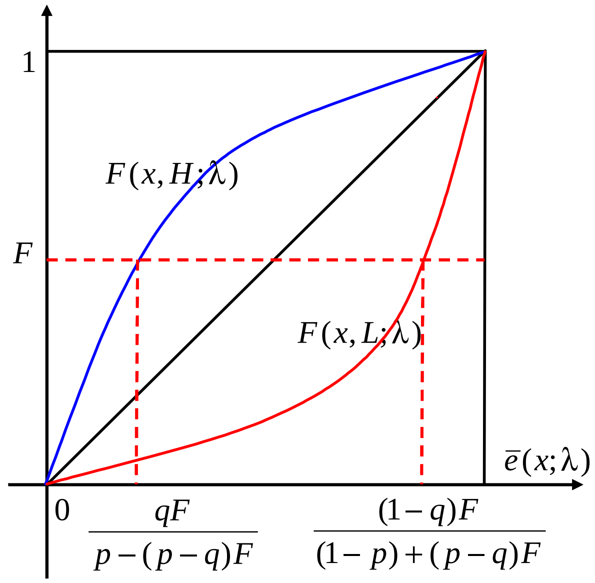

Figure 1.5

If the principal perfectly predicts what the agent is going to do beforehand ($\bar{e}(x;\lambda) = 0$ or 1), then no matter what the true output turns out to be, she will not update her prior belief due to $q, p \in (0,1)$[32]. This is what happens when the agent uses a pure strategy as in Fuchs (2007) according to Proposition 1.1. But if the agent uses a mixed strategy as in this paper, the true output carries useful information for the principal to infer whether the agent has shirked or exerted effort. As shown in Figure 1.5, $F(x;L;\lambda)$ and $F(x;H;\lambda)$ are not equal for $\bar{e}(x;\lambda) \in (0,1)$.

Moreover, the principal is always less (more) convinced of the agent having exerted effort after observing the true output to be low (high), because $F(x;L;\lambda)$ ($F(x;H;\lambda)$) is below (above) the 45 degree line. Furthermore, the convexity of $F(x;L;\lambda)$ and the concavity of $F(x;H;\lambda)$ in $\bar{e}(x;\lambda)$ imply that the principal relies more heavily on the true output in current period to infer whether the agent has shirked or exerted effort, when her history (combined with the agent's strategy) tells her less about what the agent is going to do beforehand, in the sense that $\bar{e}(x;\lambda)$ is away from 0 and 1. Therefore, the distribution of $\bar{e}(x;\lambda)$ on $[0,1]$ across the principal's possible histories measures how private the agent's efforts are to the principal.

Proposition 1.5 *Suppose that e is an effort function supporting truth-reporting.*

[32]If $p = 1$ and the principal believes with certainty that the agent is going to exert effort beforehand, then she knows she was wrong when the true output turns out to be low.

Then,

$$e_{H,0} - e_{L,0} \leq e_{H,1} - e_{L,1} \text{ and } e_{L,1} < e_{H,1}.$$

Proof. See Appendix 1.5.9. ■

Intuitively, in order to have incentives to be truthful, the principal must be convinced of being in the situation where it is in her best interest to be truthful. Because the principal is less (more) convinced of the agent having exerted effort after observing the true output to be low (high), she must prefer reporting the low (high) output when the agent has shirked (exerted effort). Hence, I have

$$\mathbf{e}_{H,0} - \mathbf{e}_{L,0} \leq y \leq \mathbf{e}_{H,1} - \mathbf{e}_{L,1}. \tag{22}$$

Then, Proposition 1.5 follows (19) immediately, which predicts that an employee who has exerted more effort is more sensitive to what one's performance evaluation turns out to be. Moreover, $e_{L,1} < e_{H,1}$ is consistent with empirical studies such as Greenberg (1986), who shows that an employee is more likely to regard a positive evaluation as being fair than a negative one.

In order to further characterize the set of the principal's beliefs, $\mathbb{N}(\lambda)$, and conduct the efficiency analysis, I have to be specific about the starting condition ρ. Formally, I define $\rho = \boldsymbol{\rho}$, where $\boldsymbol{\rho}^\top \equiv (\boldsymbol{\rho}_{L,0}\ \boldsymbol{\rho}_{L,1}\ \boldsymbol{\rho}_{H,0}\ \boldsymbol{\rho}_{H,1})^\top$ is the stationary distribution defined by

$$\boldsymbol{\rho}^\top = \boldsymbol{\rho}^\top \mathbb{Z}. \tag{23}$$

In words, Nature starts the game in the way like it has been going on forever. By algebra, I have

$$\boldsymbol{\rho} = \left(\frac{(1-q)(1-e_1)}{1-(e_1-e_0)}\ \frac{(1-p)e_0}{1-(e_1-e_0)}\ \frac{q(1-e_1)}{1-(e_1-e_0)}\ \frac{pe_0}{1-(e_1-e_0)} \right)$$

which implies that $\boldsymbol{\rho}^\top \mathbf{e} = \boldsymbol{\rho}^\top e = \frac{e_0}{1-(e_1-e_0)}$. Hence, the agent's effort functions are ranked by $\frac{e_0}{1-(e_1-e_0)}$. Intuitively, $\frac{e_0}{1-(e_1-e_0)}$ is the unconditional probability of the agent exerting effort after the game has been going on for many periods, independent of the agent's initial state.

Proposition 1.6 below explores the possibility of providing incentives more efficiently (in terms of $\frac{e_0}{1-(e_1-e_0)}$ attainable) for the principal to be truthful, without having to deal with the principal's beliefs explicitly. But the answer is negative.

Following Ely, Hörner and Olszewski (2005), I say effort function e supporting truth-reporting is belief-free if given e, the principal has incentives to be truthful regardless of her belief. Hence, when I call effort function e belief-free, it does not refer to that the agent's strategy is belief-free given the principal's truth-reporting strategy, which is true according to Lemma 1. Instead, I refer to that given e, the principal's truth-reporting strategy is belief-free.

Suppose the principal believes with certainty of the agent having shirked. Then she reports truthfully if and only if $\mathbf{e}_{H,0} - \mathbf{e}_{L,0} = y$ due to (20) and (21). Suppose the principal believes with certainty of the agent having exerted effort. Then she reports truthfully if and only if $\mathbf{e}_{H,1} - \mathbf{e}_{L,1} = y$ due to (20) and (21). So e supporting truth-reporting is belief-free if and only if

$$\mathbf{e}_{H,0} - \mathbf{e}_{L,0} = y = \mathbf{e}_{H,1} - \mathbf{e}_{L,1}$$

which does not necessarily imply that $e_{L,0} = e_{L,1}$ and $e_{H,0} = e_{H,1}$ as in perfect public equilibria.

Proposition 1.6 *Suppose that e^* is the optimal belief-free effort function supporting truth-reporting. Then,*

$$e^*_{L,0} = e^*_{L,1} = 1 - \frac{1}{\beta}\frac{c}{(p-q)^2(\theta_H - \theta_L)} \text{ and } e^*_{H,0} = e^*_{H,1} = 1$$

for $\beta \in \left[\frac{c}{(p-q)^2(\theta_H-\theta_L)}, 1\right)$.

Proof. See Appendix 1.5.10. ■

Notice that e^* is identical to the optimal perfect public equilibrium in Theorem 1.2, but with the different measure for the NED probability of the agent exerting effort

$$\frac{(p-q)^2(\theta_H - \theta_L) - (1-q)\frac{c}{\beta}}{(p-q)[(p-q)(\theta_H - \theta_L) - \frac{c}{\beta}]}\text{[33]} \tag{24}$$

which is smaller due to the different starting conditions. Specifically, the implicit starting condition in Theorem 1.2 is $(0\ 0\ x\ 1-x)^\top$ for some $x \in [0,1]$, which is different from $\boldsymbol{\rho}^*$ in which the agent's initial state can be $(L,0)$ and $(L,1)$. In order to conduct the meaningful efficiency analysis, I still use (24) to measure the NED probability of the agent exerting effort for e^*[34]. More importantly, Proposition 1.6 suggests that in order to provide incentives more efficiently for the principal to be truthful, I have to consider belief-based effort functions supporting truth-reporting with $\mathbf{e}_{H,0} - \mathbf{e}_{L,0} < \mathbf{e}_{H,1} - \mathbf{e}_{L,1}$ according to Proposition 1.5.

For a belief-based effort function e supporting truth-reporting, (20) and (21) can

[33]As the discount factor goes to one, the difference disappears.
[34]The starting condition does not affect the principal's incentives in belief-free equilibria, but it does in belief-based equilibria I will consider later.

be rewritten as

$$F(x, L; \lambda) \leq \mathbf{F} \leq F(x, H; \lambda)\ \forall \lambda \in \Lambda \text{ and } x \in \mathbb{N}(\lambda) \tag{25}$$

with

$$\mathbf{F} = \frac{y - (\mathbf{e}_{H,0} - \mathbf{e}_{L,0})}{(\mathbf{e}_{H,1} - \mathbf{e}_{L,1}) - (\mathbf{e}_{H,0} - \mathbf{e}_{L,0})} \tag{26}$$

due to $\mathbf{e}_{H,0} - \mathbf{e}_{L,0} < \mathbf{e}_{H,1} - \mathbf{e}_{L,1}$. That implies that whenever the principal observes the true output to be low (high), she assigns a probability lower (higher) than the threshold value $\mathbf{F}$ to the agent having exerted effort. Notice that if the principal is at state λ, and assigns probability x to the agent being at state $(\lambda, 1)$, then

$$F(x, L; \lambda_t) \leq \bar{e}(x; \lambda_t) \leq F(x, H; \lambda_t).$$

Hence, (25) basically requires that the separation to be uniform across the principal's possible histories. Or, in words, no matter what happened in the past, the agent's output sends the "right" message for the principal about whether the agent has shirked or exerted effort. I show that the uniform separation property as described above imposes a restriction on the agent's effort functions, which causes a non-negligible efficiency loss.

I first show in Proposition 1.7 below that the optimal belief-free effort function e^* cannot be approximated by a sequence of belief-based effort functions.

Proposition 1.7 *There exists $\varepsilon > 0$ such that no belief-based effort function supporting truth-reporting belongs to the ε-neighborhood of e^* in the space $[0, 1]^4$.*

Proof. Appendix 1.5.11. ■

By treating each effort function as a point in the space $[0, 1]^4$, Proposition 1.7 suggests that e^* is isolated from the set of belief-based effort functions supporting truth-reporting. As argued above, for each belief-based effort function supporting truth-reporting, there exists a threshold such that whenever the principal observes the true output to be low (high), she is convinced of the agent having exerted effort for a probability less (greater) than that threshold. But given e^*, the principal at state H believes with certainty of the agent having exerted effort, regardless of the true output because $e^*_{H,0} = e^*_{H,1} = 1$. At the same time, the principal at state L believes of the agent having exerted effort for probability $\frac{e^*_{L,0}p}{(1-e^*_{L,0})q+e^*_{L,0}p} = \frac{e^*_{L,1}p}{(1-e^*_{L,1})q+e^*_{L,1}p} < 1$ even after observing the true output to be high. Proposition 1.7 follows by realizing that F is a continuous function.

Therefore, e^* can not be approximated by a sequence of belief-based effort functions supporting truth-reporting. A related question considered by Mailath and Morris (2002) is that under what conditions, a perfect public equilibrium in a repeated game with public monitoring is a sequential equilibrium in an arbitrarily closed repeated game with private monitoring[35]. They require each player to strictly prefer one's action in the perfect public equilibrium than alternatives. Hence, as long as the disturbance introduced is uniformly small enough, each player has incentives to stick to one's strategy in the perfect public equilibrium. Proposition 1.7 addresses a different question which is that given the same repeated game with both public and private monitoring, whether the optimal belief-free effort function (it happens to be public) can be approximated by a sequence of belief-based effort functions supporting truth-reporting. Notice that given e^*, the principal is indifferent between reporting the low and high outputs such that the principal's truth-reporting strategy is not robust to even an arbitrarily small disturbance.

Proposition 1.7 also has an important implication on the effectiveness of numerical algorithms. Notice the Borel measure of the set of belief-free effort functions supporting truth-reporting is zero because $\mathbf{e}_{H,0} - \mathbf{e}_{L,0} = y = \mathbf{e}_{H,1} - \mathbf{e}_{L,1}$, which implies that the set is two-dimensional in the four-dimensional space $[0,1]^4$. Therefore, e^*, the potential optimal effort function supporting truth-reporting, can not be approximated by some common numerical algorithms. For instance, consider a common numerical algorithm as follows: let the interval $[0,1]$ be discretized as $\Delta = \left\{\frac{0}{N}, \frac{1}{N}, \cdots, \frac{N}{N}\right\}$ where $N \geq 1$ is an integer. For each $e \in \Delta^4$, calculate $\mathbb{N}(\lambda)$ by (16) and (17)[36]. Then I check the principal's incentives by (20) and (21). If e supports truth-reporting, calculate $\frac{e_0}{1-(e_1-e_0)}$. As N goes to infinity, I expect the result generated by this numerical algorithm to converge to the optimal effort function supporting truth-reporting. But if e^* is optimal, which happens frequently in my computation, it is impossible to get close to e^* no matter how large N becomes.

Now I start to identify the efficiency loss associated with belief-based effort functions. Given $\mathbf{F} \in [0,1]$, I have

$$\overline{e}(x;\lambda) \in \left[\frac{q\mathbf{F}}{p-(p-q)\mathbf{F}}, \frac{(1-q)\mathbf{F}}{(1-p)+(p-q)\mathbf{F}}\right] \equiv \Omega(\mathbf{F})\ \forall \lambda \in \Lambda \text{ and } x \in \mathbb{N}(\lambda) \quad (27)$$

by (25), which implies that the principal expects the agent to exert effort in current period with a probability greater than $\frac{q\mathbf{F}}{p-(p-q)\mathbf{F}}$ and less than $\frac{(1-q)\mathbf{F}}{(1-p)+(p-q)\mathbf{F}}$ under any

[35]In the sense that the probability of both players receiving the same private signal is arbitrarily close to one.

[36]It is feasible because we just have to iterate on the two extreme points as shown by Phelan and Skrzypacz (2008).

circumstances. Therefore,

$$(1 - F(x, \theta; \lambda))\mathbf{e}_{\vartheta,0} + F(x, \theta; \lambda)\mathbf{e}_{\vartheta,1} \in \Omega(\mathbf{F}) \ \forall \lambda \in \Lambda,\ x \in \mathbb{N}(\lambda) \text{ and } \vartheta, \theta \in \Theta\text{[37]}$$

which implies that the principal expects the agent to exert effort (on average) in the future for a probability bounded by $\Omega(\mathbf{F})$, regardless of her history, the reported output and the true output. The reason is that by (17), $F(x, \theta; \lambda) \in \mathbb{N}(\vartheta)$ such that the principal expects the agent to exert effort for probability

$$(1 - F(x, \theta; \lambda))e_{\vartheta,0} + F(x, \theta; \lambda)e_{\vartheta,1} \in \Omega(\mathbf{F}).$$

due to (27). This is also true in the following periods. However, (20) and (21) require

$$0 < y \leq \frac{(1-q)\mathbf{F}}{(1-p) + (p-q)\mathbf{F}} - \frac{q\mathbf{F}}{p - (p-q)\mathbf{F}},$$

or equivalently,

$$(p-q)[1-(p-q)y]\mathbf{F}^2 + (p-q)[(2p-1)y - 1]\mathbf{F} + p(1-p)y \leq 0. \qquad (28)$$

Otherwise, the principal does not have incentives to report the high output under any circumstances. Furthermore, (28) implies $\mathbf{F} \in \left[\underline{\mathbf{F}}, \overline{\mathbf{F}}\right]$ with

$$\frac{-(p-q)[(2p-1)y-1] - \sqrt{(p-q)^2[(2p-1)y-1]^2 - 4p(1-p)(p-q)[1-(p-q)y]y}}{2(p-q)[1-(p-q)y]}$$

$$\frac{-(p-q)[(2p-1)y-1] + \sqrt{(p-q)^2[(2p-1)y-1]^2 - 4p(1-p)(p-q)[1-(p-q)y]y}}{2(p-q)[1-(p-q)y]}$$

as $\underline{\mathbf{F}}$ and $\overline{\mathbf{F}}$ respectively where

$$(p-q)^2[(2p-1)y-1]^2 - 4p(1-p)(p-q)[1-(p-q)y]y \geq 0\text{[38]}. \qquad (29)$$

The idea can be clearly illustrated in Figure 1.5. As $\mathbf{F}$ goes to 1, $\Omega(\mathbf{F})$ becomes smaller due to the convexity of $F(x, L; \lambda)$ and the concavity of $F(x, H; \lambda)$ in $\overline{e}(x; \lambda)$. So the potential impact of the principal's reported output becomes smaller on the NED probability of the agent exerting effort in the future, as measured by the length of $\Omega(\mathbf{F})$. At some point, it becomes too small to be able to compensate the cost of reporting the high output for the principal.

As a result, either $e_{H,0}$ or $e_{H,1}$ must be less than $\frac{(1-q)\overline{\mathbf{F}}}{(1-p)+(p-q)\overline{\mathbf{F}}} < 1$, otherwise

[37]It does not necessarily imply $\mathbf{e}_\gamma \in \Omega(\mathbf{F})\ \forall \gamma$.

[38]Otherwise, there exists no belief-based effort function supporting truth-reporting.

$\overline{e}(x;H) > \frac{(1-q)\overline{\mathbf{F}}}{(1-p)+(p-q)\overline{\mathbf{F}}}$ $\forall x \in [0,1]$ which contradicts (27). This features an efficiency loss when compared to e^* with $e^*_{H,0} = e^*_{H,1} = 1$. To illustrate how large it can be, consider an example with $q = 0.3$, $p = 0.7$, $\theta_L = 0$, $\theta_H = 8$, $c = 1$ and $\beta = 0.85$. By calculation, I have $\frac{(1-q)\overline{\mathbf{F}}}{(1-p)+(p-q)\overline{\mathbf{F}}} \doteq 0.94$ which implies that the agent has to lower the probability of exerting effort at either state $(H,0)$ or state $(H,1)$ by at least 6 percents, which turns out to be a quiet underestimate when it comes to computation.

Proposition 1.8 *Suppose $\rho = \boldsymbol{\rho}$. Then,*

(a) $\mathbb{N}_{t+1}(L) = \mathbb{N}_{t+1}(H) \equiv \mathbb{N}_{t+1}$;

(b) $\inf \mathbb{N}(L) = \inf \mathbb{N}(H) = \inf \bigcup_{t=1}^{\infty} \mathbb{N}_t \equiv \underline{x}$ and $\sup \mathbb{N}(L) = \sup \mathbb{N}(H) = \sup \bigcup_{t=1}^{\infty} \mathbb{N}_t \equiv \overline{x}$;

(c) in Lemma 2, it suffices to check (20) and (21) $\forall \lambda \in \Lambda$ and $x \in \{\underline{x}, \overline{x}\}$.

Proof. See Appendix 1.5.12. ■

(a) does not requires $\rho = \boldsymbol{\rho}$. It follows the fact that given the principal's history $(\vartheta^{t+1}, \theta^{t+1})$, the principal's belief is determined by $(\vartheta^{t+1}, \theta^{t})$ while her state in period $t+1$ is determined by θ_t. And (b) follows (a) and the fact that the principal's initial belief $\frac{\rho_{\lambda,1}}{\rho_{\lambda,0}+\rho_{\lambda,1}} \in \mathbb{N}_0(\lambda)$ will not be the extreme point due to $\rho = \boldsymbol{\rho}$. The agent's effort is more private, if $\underline{x}$ is more away from 0 and $\overline{x}$ is more away from 1. Finally, (c) follows the convexity of $F(x, L; \lambda)$ and the concavity of $F(x, H; \lambda)$ in $\overline{e}(x; \lambda)$.

Theorem 1.3 shows that hiding the agent's efforts from the principal by mixed strategies incurs a non-negligible efficiency loss.

Theorem 1.3 *There exists $\varepsilon > 0$ such that if e is a belief-based effort function supporting truth-reporting, then the principal expects the agent to exert effort for a probability less than $1-\varepsilon$ regardless of her history. In addition,*

$$\eta < \underline{x} < \overline{x} < 1-\eta \text{ for some } \eta > 0.$$

Furthermore, if e is strictly more efficient than e^*, then

$$e_{L,0} > e_{L,1}.$$

Proof. See Appendix 1.5.13. ■

The first result follows by setting $\varepsilon = 1 - \frac{(1-q)\overline{\mathbf{F}}}{(1-p)+(p-q)\overline{\mathbf{F}}}$ as shown above. It is a non-negligible efficiency loss in the sense that given the agent's optimal belief-free strategy, the principal expects the agent to exert effort with probability one if the reported output is high in the prior period. However, given any belief-based strategy for the agent, the principal can never expect the agent to exert effort for a probability greater than $1-\varepsilon$ under any circumstances.

In addition, $\eta < \underline{x} < \overline{x} < 1 - \eta$ implies that the agent's effort has to be private enough in the sense that the principal is unable to infer it with certainty under any circumstances. Otherwise, there exist a initial state and a sequence of one-period histories $\{(\vartheta_t, \theta_t)\}_{t=0}^{\infty}$, along which the principal is going to be convinced at some point, or eventually with certainty of the agent having shirked or exerted effort.

If $e_{L,0} < e_{L,1}$, the sequence of the principal's beliefs along history $\{(\vartheta_t = L, \theta_t = L)\}_{t=0}^{\infty}$ is monotonic and convergent. In other words, no matter what the principal's initial belief x_0 is, she will be convinced eventually of the agent having exerted effort for probability x_∞, as shown in Figure 1.6a for the case $x_0 > x_\infty$. And if $e_{H,0} < e_{H,1}$, the sequence of the principal's beliefs along history $\{(\vartheta_t = H, \theta_t = H)\}_{t=0}^{\infty}$ is monotonic and convergent too, as shown in Figure 1.6b for the case $x_0 < x_\infty$.

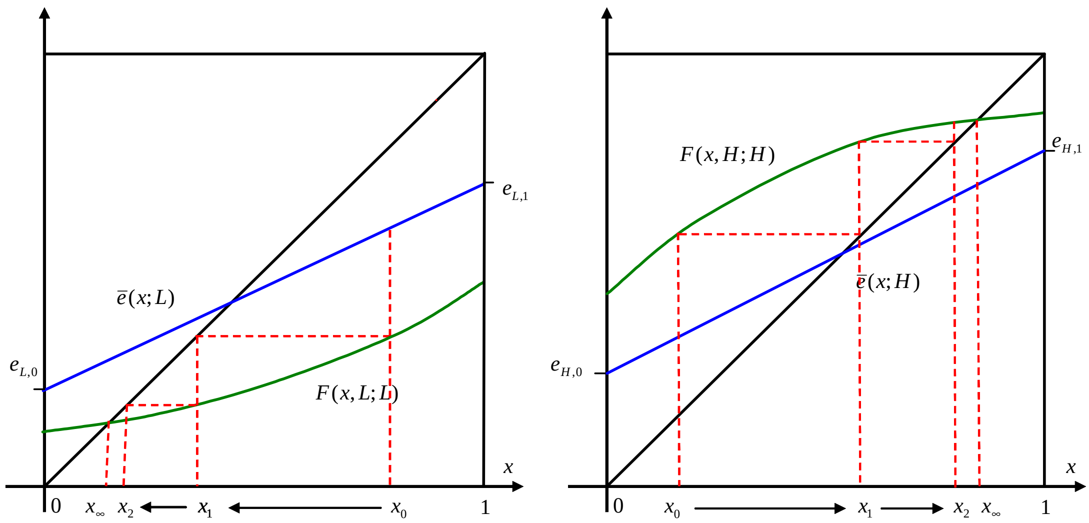

Figure 1.6a and 1.6b

Furthermore, if $e_{L,0} = 0$, the principal is going to be convinced eventually with certainty of the agent having shirked due to $x_\infty = 0$, and if $e_{H,1} = 1$, the principal is going to be convinced eventually with certainty otherwise due to $x_\infty = 1$. No matter the principal is convinced with certainty of one way or another, she reports truthfully only if she is indeed indifferent between reporting the low and high outputs as shown before. That is why it is important for the agent to keep his effort private under any circumstances, because the extreme beliefs across the principal's possible histories, $\underline{x}$ and $\overline{x}$, determine the principal's incentives to be truthful according to (c) in Proposition 1.8.

To summarize, in a belief-based effort function supporting truth-reporting, the agent has to avoid punishing (rewarding) the principal to the full extent by shirking (exerting effort) with certainty, as he does in the optimal belief-free (public) effort function. The reason is that by shirking (exerting effort) with more certainty, the agent makes his effort more predictable by the principal, therefore less useful in providing incentives for the principal to be truthful.

Furthermore, Theorem 1.3 shows that the agent responds to the reported low output more positively with $e_{L,0} > e_{L,1}$, if he shirked. This is consistent with empirical studies such as Greenberg who shows that an employee regards a performance evaluation match his self-evaluation as being fair, and exerts more effort in the future.

Proposition 1.9 below is devoted to measure how private the agent's effort is by calculating $\underline{x}$ and $\overline{x}$, given his effort function.

Proposition 1.9 *Suppose that e is an effort function with*

$$e_{L,0} \geq e_{L,1},\ e_{L,0} \leq e_{H,0} \text{ and } e_{H,0} - e_{L,0} \leq e_{H,1} - e_{L,1}.$$

Then, (a) $\underline{x} = F(\overline{x}, L; L)$ *and* $\overline{x} = F(\underline{x}, H; H)$ *if* $e_{H,0} \geq e_{H,1}$*;*
(b) $\underline{x} = F(\overline{x}, L; L)$ *and* $\overline{x} = F(\overline{x}, H; H)$ *if* $e_{H,0} \leq e_{H,1}$*.*

Proof. See Appendix 1.5.14. ■

The only restrictive assumption is $e_{L,0} \leq e_{H,0}$, because $e_{L,0} \geq e_{L,1}$ is necessary for belief-based effort functions supporting truth-reporting to be strictly more efficient than e^* according to Theorem 1.3, and $e_{H,0} - e_{L,0} \leq e_{H,1} - e_{L,1}$ follows Proposition 1.5. Moreover, $e_{L,0} \leq e_{H,0}$ simply implies that no matter what the principal's belief is, she expects the agent to exert more effort following the reported high output because

$$\overline{e}(x; L) = e_{L,0} + (e_{L,1} - e_{L,0})x \leq e_{H,0} + (e_{H,1} - e_{H,0})x = \overline{e}(x; L)\ \forall x \in [0, 1]$$

where $e_{L,1} - e_{L,0} \leq e_{H,1} - e_{H,0}$ is implied by $e_{H,0} - e_{L,0} \leq e_{H,1} - e_{L,1}$. It makes sense for three reasons: first, the agent has to compensate the principal for reporting the high output in general because it is costly for the principal; second, it is consistent with empirical studies such as Greenberg (1986) who shows that an employee is more likely to regard a positive evaluation as being fair than a negative one; third, it is a robust feature of the optimal effort function supporting truth-reporting in my numerical analysis.

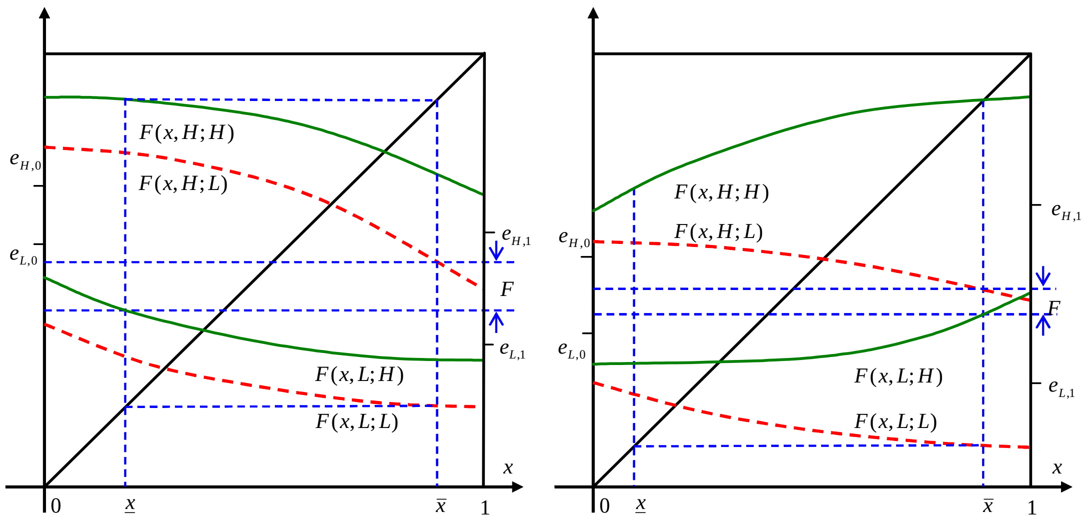

Figure 1.7a and 1.7b

The case (a) in Proposition 1.9 is illustrated in Figure 1.7a. After observing the true output to be low, the principal assigns a probability ranging from $F(\overline{x}, L; L)$ to $F(\underline{x}, L; H)$ to the agent having exerted effort. And after observing the true output to be high, the principal assigns a probability ranging from $F(\overline{x}, H; L)$ to $F(\underline{x}, H; H)$ to the agent having exerted effort. Therefore, e supports truth-reporting if

$$F(\underline{x}, L; H) \leq \mathbf{F} \leq F(\overline{x}, H; L).$$

Suppose that the principal's belief in period 0 is x_0, which belongs to $[\underline{x}, \overline{x}]$ due to $\rho = \boldsymbol{\rho}$. What is the history along which the principal is able to get more information about the agent's effort as fast as possible, or equivalent, the principal's belief converges to $\underline{x}$ or $\overline{x}$ as fast as possible? The answer is

$$\{(\vartheta_t = L, \theta_t = H), (\vartheta_{t+1} = H, \theta_{t+1} = L)\} \text{ for } t = 0, 2, \cdots$$

which features an oscillation off the equilibrium path.

The case (b) in Proposition 1.9 is illustrated in Figure 1.7b. After observing the true output to be low, the principal assigns a probability ranging from $F(\overline{x}, L; L)$ to $F(\overline{x}, L; H)$ to the agent having exerted effort. And after observing the true output to be high, the principal assigns a probability ranging from $F(\overline{x}, H; L)$ to $F(\overline{x}, H; H)$ to the agent having exerted effort. Therefore, e supports truth-reporting if

$$F(\overline{x}, L; H) \leq \mathbf{F} \leq F(\overline{x}, H; L).$$

Suppose that the principal's belief in period 0 is x_0, which belongs to $[\underline{x}, \overline{x}]$ due to $\rho = \boldsymbol{\rho}$. What is the history along which the principal is able to get more information about the agent's effort as fast as possible, or equivalent, the principal's belief converges to $\underline{x}$ or $\overline{x}$ as fast as possible? The answer is

$$\{(\vartheta_t = H, \theta_t = H)\}_{t=0}^{\infty}$$

which features a monotonic convergence on the equilibrium path. However, in order to reach $\underline{x}$, I should add $\{(L, H), (L, L)\}$ at the end of a very long sequence of (H, H).

Therefore, in some sense, effort function e as in the case (a) reveal more information to the principal who is engaging on misreporting. That is not consistent with our goal of providing incentives more efficiently for the principal to be truthful. Lemma 3 provides a partial result about why effort function e, as in the case (a), is not likely to be optimal. But the argument here provides an intuitive explanation.

Define

$$\mathbb{Q}(\theta) \equiv \{x \in [0,1] \,|\exists \lambda \in \Lambda \text{ and } x' \in \mathbb{N}(\lambda) \text{ such that } x = F(x', \theta; \lambda)\} \subseteq [0,1] \;\forall \theta \in \Theta \tag{30}$$

as the set of possible probabilities assigned by the principal to the agent having exerted effort, after observing the true output to be θ. Notice that $\mathbb{Q}(\theta)$ serves as a bridge between the agent and the principal in the sense that it is determined by the agent's strategy, and at the same time, it determines whether the principal has incentives to be truthful.

Apparently, (25) suggests that $\mathbb{Q}(L)$ is below and separable from $\mathbb{Q}(H)$ in the space $[0,1]$. Theorem 1.4 further shows that the distance between $\mathbb{Q}(L)$ and $\mathbb{Q}(H)$ is zero in the optimal belief-based effort function.

Theorem 1.4 *Suppose that e^+ is the optimal belief-based effort function supporting truth-reporting. If $e^+_{L,0} \leq e^+_{H,0}$ and e^+ is strictly more efficient than e^*, then*

$$\sup \mathbb{Q}^+(L) = \mathbf{F}^+ = \inf \mathbb{Q}^+(H).$$

Proof. See Appendix 1.5.15. ■

It suffices to prove $F(\underline{x}^+, L; H) = \mathbf{F}^+ = F(\overline{x}^+, H; L)$ in case (a) and $F(\overline{x}^+, L; H) = \mathbf{F}^+ = F(\overline{x}^+, H; L)$ in case (b). The distance between $\mathbb{Q}^+(L)$ and $\mathbb{Q}^+(H)$ measures how well the agent is able to keep his effort private. The reason is that the principal relies heavily on the output in the current period to infer the agent's effort, only because the agent has been successfully keeping the principal from inferring too much from her history (along with the agent's strategy) about what the agent is going to do in the current period. Therefore, $\sup \mathbb{Q}^+(L) < \inf \mathbb{Q}^+(H)$ implies that the agent keeps his effort too private to be necessary in terms of providing incentives for the principal to be truthful. Because keeping the agent's effort private involves the efficiency loss, it is not optimal to overdo it. Specifically, if $\sup \mathbb{Q}^+(L) < \inf \mathbb{Q}^+(H)$, the agent is able to exert effort with a slightly larger probability at each state, while still keeping his effort private enough in terms of providing incentives for the principal to be truthful.

Effort functions as case (b) have all the features consistent with empirical studies on how employees respond to subjective performance evaluations. So Lemma 3 tries to shed some light on why it is likely to be optimal, by showing that $e^+_{H,0} \geq e^+_{H,1}$ is likely to bind.

Lemma 1.3 *Suppose $e^+_{L,0} \leq e^+_{H,0}$, $e^+_{H,0} \geq e^+_{H,1}$ and e^+ is strictly more efficient than e^*. Then, if $p(1-q)^2 \geq q^2(1-p)$*[39]*, one of the following statements must be true: (a) $e^+_{L,0} = e^+_{H,0}$; (b) $e^+_{L,1} = 0$; (c) $e^+_{H,0} = e^+_{H,1}$.*

Proof. See Appendix 1.5.16. ■

Finally, I try to measure numerically the efficiency outcome associated with the agent's belief-based effort functions. Given $q = 0.01$, $\theta_L = 0$, $\theta_H = 6$, $c = 2$ and $\beta = 0.65$, I have

	$e^+_{L,0}$	$e^+_{L,1}$	$e^+_{H,0}$	$e^+_{H,1}$	$\boldsymbol{\rho}^{+\top}\mathbf{e}^+$	$\boldsymbol{\rho}^{*\top}\mathbf{e}^*$	Productive	(a) or (b)
$p = .78$	.113	.108	.976	.976	.361	.430	NO	(b)
$p = .80$	.190	.117	.980	.981	.508	.532	NO	(b)
$p = .82$	.249	.156	.982	.988	.613	.617	NO	(b)
$p = .84$	.340	.139	.988	.988	.701	.688	YES	(b)
$p = .86$	.421	.141	.990	.990	.768	.750	YES	(b)
$p = .88$	.500	.147	.992	.992	.822	.802	YES	(b)

[39]It can be easily satisfied with $q < \frac{1}{2} < p$.

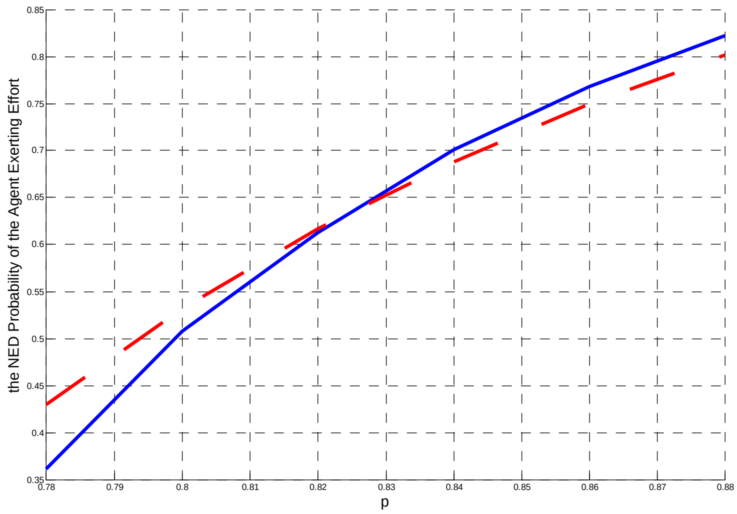

Figure 1.8

in which the blue solid line represents $\boldsymbol{\rho}^{+\top}\mathbf{e}^{+}$ and the red dashed line represents $\boldsymbol{\rho}^{*\top}\mathbf{e}^{*}$.

Given q, if p becomes larger, the agent's effort becomes more informative about the true output observed by the principal. Hence, the potential efficiency gain is larger as well by using the agent's effort to provide incentives for the principal to be truthful. Also, $F(x, L; \lambda)$ and $F(x, H; \lambda)$ become further away from the 45 degree line, as shown in Figure 1.5, because the true output is more informative for the principal about the agent's effort. Therefore, $\mathbf{F}$ can be potentially closer to 1 so that the potential loss from hiding the agent's effort from the principal becomes smaller, as measured by $1 - \frac{(1-q)\overline{\mathbf{F}}}{(1-p)+(p-q)\overline{\mathbf{F}}}$. Both factors contribute to what is shown in Figure 1.8.

I make two assumptions in this section: the agent's strategy depends on the one-period history, and the principal reports truthfully even off the equilibrium path. In next section, I try to relax them.

1.3.3 Sequential Equilibria

In this section, I consider sequential equilibria in which the principal reports truthfully on the equilibrium path, without imposing any constraints on the agent's strategies. The contract is w^{**} according to Lemma 1. Therefore, the principal's belief in period t has the support E^t, which grows over time. However, the agent's belief is simple in the sense that he always assigns probability one to the principal being on the equilibrium path, therefore being going to be truthful in the future. Hence, the principal's strategy is public and static in the agent's point of view even though she may report untruthfully off the equilibrium path. By Definition 1, the agent's continuation strategy following any possible history (ϑ^t, e^t) is the best response to the principal's same truth-reporting strategy. By mixing the agent's continuation strategy following (ϑ^t, e^t) with probability $\mu[\vartheta^t, \theta^t = \vartheta^t](e^t)$ assigned by the principal on the equilibrium path, a new SE with respect to w^{**} featuring truth-reporting can be constructed. The reason is that the principal with history $(\vartheta^t, \theta^t = \vartheta^t)$ has incentives to be truthful when he believes of the agent using continuation strategy $e \mid (\vartheta^t, e^t)$ with probability $\mu[\vartheta^t, \theta^t = \vartheta^t](e^t)$ $\forall e^t$. Given that, a partial recursive structure is able to be established so that a lower and an upper bound on the efficiency attainable are derived in Proposition 1.10.

Define a mapping $\Psi : 2^{[0,1]} \to 2^{[0,1]}$ as

$$\Psi(X) \equiv \left\{ x \in [0,1] \,\middle|\, \exists e, \mathbf{e}_{L,0}, \mathbf{e}_{L,1}, \mathbf{e}_{H,0}, \mathbf{e}_{H,1} \in [0,1] \text{ such that (32)-(36).} \right\} \tag{31}$$

where

$$x = (1-\beta)e + \beta \begin{bmatrix} (1-e)(1-q) \\ e(1-p) \\ (1-e)q \\ ep \end{bmatrix}^\top \begin{bmatrix} \mathbf{e}_{L,0} \\ \mathbf{e}_{L,1} \\ \mathbf{e}_{H,0} \\ \mathbf{e}_{H,1} \end{bmatrix} \tag{32}$$

$$\frac{(1-e)(1-q)}{(1-e)(1-q)+e(1-p)}(\mathbf{e}_{H,0}-\mathbf{e}_{L,0}) + \frac{e(1-p)}{(1-e)(1-q)+e(1-p)}(\mathbf{e}_{H,1}-\mathbf{e}_{L,1}) \le y \tag{33}$$

$$\frac{(1-e)q}{(1-e)q+ep}(\mathbf{e}_{H,0}-\mathbf{e}_{L,0}) + \frac{ep}{(1-e)q+ep}(\mathbf{e}_{H,1}-\mathbf{e}_{L,1}) \ge y \tag{34}$$

$$\frac{(1-e)(1-q)}{(1-e)(1-q)+e(1-p)}\mathbf{e}_{L,0} + \frac{e(1-p)}{(1-e)(1-q)+e(1-p)}\mathbf{e}_{L,1} \in co(X). \tag{35}$$

$$\frac{(1-e)q}{(1-e)q+ep}\mathbf{e}_{H,0} + \frac{ep}{(1-e)q+ep}\mathbf{e}_{H,1} \in co(X) \tag{36}$$

$\forall \beta \in \left[\frac{c}{(p-q)^2(\theta_H-\theta_L)+[1-(p-q)]c}, 1\right)$ (the set is chosen so that $y \in [0,1]$). Proposition 1.10 shows that the mapping Ψ is able to generate a lower and an upper bound on NED probability of the agent exerting effort attainable by SE with respect to w^{**} in which the principal reports truthfully.

Proposition 1.10 *Suppose that (e, r) is a SE with respect to w^{**} in which the principal's strategy features truth-reporting. Then,*

$$(1-\beta)\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t e_t \,\middle|\, s\right] \in \Psi^\infty([0,1]).$$

Proof. See Appendix Q. ■

Theoretically, the same logic applies in any continuation game. And the estimation is able to become more and more precise. However, it is not practical. For instance, if I consider a continuation game starting from period 2, there are 16 instead of 4 NED probabilities of the agent exerting effort consistent with the number of the agent's possible two-period histories.

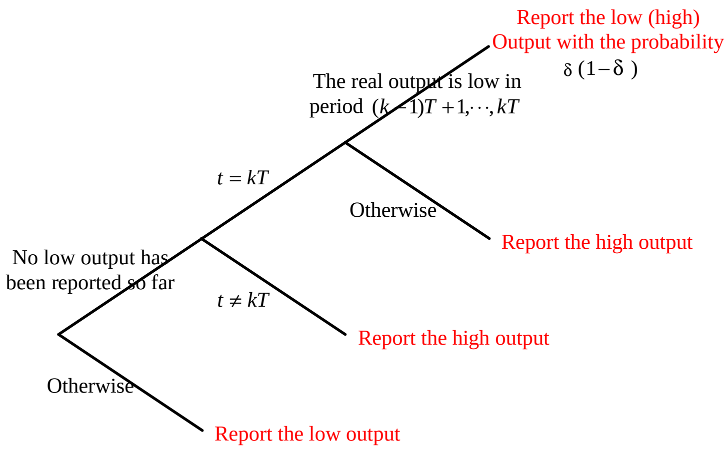

Figure 1.9

Furthermore, I replicate the T-period review equilibrium in Fuchs (2007) without referring to termination as follows: let the agent's strategy be to exert effort in period t if and only if no low output has been reported before[40], and the principal's review strategy be illustrated in Figure 1.9 below where $k = 1, 2, \cdots$ and T is the length of review phase. Then this is a SE with respect to w^* if

$$\frac{\beta^T \delta(1-p)^{T-1}(p-q)(1-\beta^T)}{[1-\beta^T(1-\delta(1-p)^T)](1-\beta)} \geq \frac{c}{(p-q)(\theta_H-\theta_L)-c}[41]. \qquad (37)$$

Notice that only one high output out of T outputs is enough for the principal to report the high output with probability one in period T. Consequently, the agent off the equilibrium path has stronger incentives to exert effort because the probability of him having no high output so far is higher. For the same reason, the agent on the equilibrium path has weakest incentives to exert effort in period T. Hence, it suffices to check the incentive compatibility constraint for the agent in period T who has exerted effort for the first $T-1$ periods as shown in (37).

[40]The agent's strategy is pure and public.
[41]The optimal PPE in Theorem 1 is a special case with $T = 1$ and the minimum δ solving (37).

Given $q = 0.33$, $p = 0.67$, $\theta_L = -4$, $\theta_H = 8$, $c = 2$ and $\beta \in [0.63, 0.99]$, I have

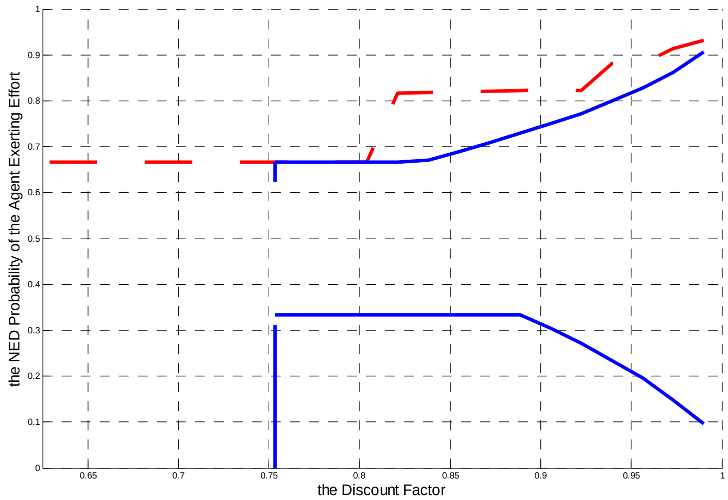

Figure 1.10

in which the red line represents the efficiency attainable by the T-period review equilibrium, and the blue lines represent the lower and upper bound on the efficiency attainable by SE in which the principal reports truthfully. The comparison is interesting because in the T-period review equilibrium, the principal is the only party keeping private information on the equilibrium path, while the agent is the only party keeping private information on the equilibrium path in SE in which the principal reports truthfully.

It seems that it is optimal for the principal to keep private information on the equilibrium path than the agent.

1.4 Conclusion

I have studied an infinitely repeated principal-agent problem in which the principal privately observes and publicly reports the agent's output. The role of the agent's private strategies, which depend on the history of his private efforts, is examined in providing incentives for the principal to be truthful. I show that there is a non-negligible efficiency loss associated with the use of the agent's private history as an incentive device. This efficiency loss may, or may not be justified by the efficiency gain. Moreover, the agent's optimal strategy is shown to be consistent with empirical studies on how employees respond to subjective performance evaluations.

1.5 Appendix

1.5.1 Proof of Proposition 1.1

I have $V_{SE}(w) \subseteq V^1_{WPBE}(w)$ because SE is a refinement of WPBE and $V^1_{WPBE}(w) \supseteq V^2_{WPBE}(w)$. So it suffices to prove $V_{SE}(w) \supseteq V^1_{WPBE}(w)$ and $V^1_{WPBE}(w) \subseteq V^2_{WPBE}(w)$. Notice $V_{SE}(w) \supseteq V^1_{WPBE}(w)$ follows immediately if the full support assumption is satisfied. But unfortunately, that is not the case in this model.

Suppose (e, r) is a WPBE with respect to w in which the agent's strategy is pure. Notice θ^t has the full support regardless of the agent's strategy due to $q, p \in (0, 1)$. Therefore, whether the principal's history (ϑ^t, θ^t) is on the equilibrium path is exclusively determined by her own strategy in the first t periods.

(1) By (4), given a public history ϑ^t, the principal with any private history θ^t assigns the probability one to the agent having the private history e^t defined recursively as $e_0 = e(\emptyset)$ and

$$e_{\tau+1} = e((\vartheta_0, \cdots, \vartheta_\tau), (e_0, \cdots, e_\tau)) \; \forall \tau = 0, \cdots, t-2$$

where e is a deterministic function because the agent uses pure strategy. Therefore, the agent's any possible history $(\vartheta^t, \overline{e}^t)$ with $\overline{e}^t \neq e^t$ is not in the support of the principal's belief at (ϑ^t, θ^t) $\forall \theta^t$.

(i) Assume (ϑ^t, θ^t) $\forall \theta^t$ is off the equilibrium path. Therefore, (ϑ^t, e^t) $\forall e^t$ is off the equilibrium path too because otherwise there exists θ^t such that (ϑ^t, θ^t) is on the equilibrium path. Let the agent shirks and the principal NED the expected transfer following ϑ^t which form a SE with respect to $w \mid \vartheta^t$. And it generates the minmax NED payoffs for both parties. As a result, the agent's optimal strategy does not change because (ϑ^t, θ^t) $\forall \theta^t$ is always off the equilibrium path given the principal's same strategy in the first t periods. And the principal's optimal strategy does not change either because she does not have incentives to put some weights on her minmax NED payoff in $w \mid \vartheta^t$.

(ii) Assume (ϑ^t, θ^t) is on the equilibrium path for $\theta^t \in \overline{\Theta}^t \subseteq \Theta^t$.

Then the agent's belief at $(\vartheta^t, \overline{e}^t)$ $\forall \overline{e}^t \neq e^t$ is defined by (3) with the same support $\overline{\Theta}^t$. So I first replace the agent's continuation strategy following $(\vartheta^t, \overline{e}^t)$ $\forall \overline{e}^t \neq e^t$ given the belief defined by (3). Then I replace the principal's continuation strategy following (ϑ^t, θ^t) $\forall \theta^t \notin \overline{\Theta}^t$ with the one following (ϑ^t, θ^t) $\forall \theta^t \in \overline{\Theta}^t$. Notice the replacement proposed here does not guarantee the principal's continuation strategy following ϑ^t is the same regardless of θ^t because $\overline{\Theta}^t$ may not be a singleton set.

Recursively applying (i) or (ii) from the beginning gives a payoff-equivalent SE with respect to w in which the agent's strategy is pure.

(2) Replace the agent's continuation strategy following $(\vartheta^t, \bar{e}^t)$ $\forall \bar{e}^t \neq e^t$ with the one following $(\vartheta^t, \bar{e}^t)$ because WPBE does not impose any restrictions on beliefs off the equilibrium path. Recursively applying (i) or the replacement proposed here from the beginning gives a payoff-equivalent WPBE with respect to w in which the agent's strategy is pure and public.

1.5.2 Proof of Proposition 1.2

Assume $w[\vartheta^t](\vartheta_t = \theta_L) > w[\vartheta^t](\vartheta_t = \theta_H)$ for some ϑ^t.

Throughout the proof, I denote ϑ_t, the reported output in period t, by θ_i and θ_j for $i, j \in \{L, H\}$ with $i \neq j$. For notational simplification, I denote (ϑ^t, θ_i) by $\langle\theta_i\rangle$ and (ϑ^t, θ_j) by $\langle\theta_j\rangle$.

Define the contract w' by altering the continuation contract $w \mid \vartheta^t$ as follows,

$$w'[\vartheta^t](\theta_i) = w[\vartheta^t](\theta_j) \text{ and } w' \mid \langle\theta_i\rangle = w \mid \langle\theta_j\rangle\,,$$

and the strategy profile $s' = (e', r')$ by altering the continuation strategies $e \mid (\vartheta^t, e^t)$ and $r \mid (\vartheta^t, \theta^t)$ as follows,

$$e'[\vartheta^t, e^t](e_t) = e[\vartheta^t, e^t](e_t) \text{ and } e' \mid (\langle\theta_i\rangle, e^t) = e \mid (\langle\theta_j\rangle, e^t) \text{ for } \forall e^t$$

$$r'[\vartheta^t, \theta^t; \theta_t](\theta_i) = r[\vartheta^t, \theta^t; \theta_t](\theta_j) \text{ and } r' \mid (\langle\theta_i\rangle, \theta^t) = r \mid (\langle\theta_j\rangle, \theta^t) \text{ for } \forall \theta^t,$$

while keeping anything else unchanged. In words, given the public history ϑ^t, the transfer in period t, as well as the continuation contract starting at period $t+1$, contingent on the reported output $\theta_L(\theta_H)$ in the contract w' are the same with the ones contingent on the reported output $\theta_H(\theta_L)$ in the contract w. In period t following the public history ϑ^t, the agent's strategy doesn't change, but the principal reports $\theta_L(\theta_H)$ whenever he is supposed to report $\theta_H(\theta_L)$ according to r. After that, the continuation strategies are chosen from s by treating the reported output $\theta_L(\theta_H)$ as $\theta_H(\theta_L)$.

I try to show (e', r') is a SE with respect to w'. It suffices to show

$$\varphi[\langle\theta_i\rangle, e^{t+1}; s'](\theta^{t+1}) = \varphi[\langle\theta_j\rangle, e^{t+1}; s](\theta^{t+1})$$

$$\mu[\langle\theta_i\rangle, \theta^{t+1}; s'](e^{t+1}) = \mu[\langle\theta_j\rangle, \theta^{t+1}; s](e^{t+1})$$

which are both true even when the agent's belief is defined by (5), therefore independent of the history of reported outputs.

That completes the proof.

1.5.3 Proof of Proposition 1.3

Define a function $B : [\underline{\theta}, +\infty) \to \mathbb{R}$ as

$$B(x) \equiv \max U^A + U^P \text{ s.t. } (U^A, U^P) \in \Gamma(\Lambda(x)) \tag{38}$$

where $\Lambda(x) \equiv \{(U^A, U^P) \mid U^A \geq w_L, U^P \geq \underline{\theta} - w_L \text{ and } U^A + U^P \leq x\}$. I show $\overline{U}(w) \leq B^\infty(\overline{\theta} - c)$ as follows: notice $\Gamma^\infty(V(w)) \subseteq V^*(w) \subseteq \Lambda(\overline{\theta} - c)$ where the first relation follows the fact $(w_L, \underline{\theta} - w_L)$ is the minmax payoff pair and the second relation follows the fact $\overline{\theta} - c$ is the maximum total payoff. Therefore, I have $\Gamma^\infty(V(w)) \subseteq \Gamma(\Lambda(\overline{\theta} - c))$ because Γ is monotonic therefore $\overline{U}(w) \leq B(\overline{\theta} - c) \leq \overline{\theta} - c$. By induction, a decreasing sequence $\left\{B^m(\overline{\theta} - c) \geq \overline{U}(w)\right\}_{m=0}^\infty$ can be constructed with $B^\infty(\overline{\theta} - c)$ as an upper bound (not necessarily the least upper bound) on $\overline{U}(w)$.

Lemma 1.4 *(a) If $(p-q)^2(\theta_H - \theta_L) \geq (1-q)c$,*

$$B(x) = \begin{cases} \beta x + (1-\beta)\underline{\theta} & , \text{ for } x \in \left[\underline{\theta}, \underline{\theta} + \frac{1-\beta}{\beta(p-q)} \max\{c, (p-q)(w_H - w_L)\}\right) \\ \beta x + (1-\beta)\left(\overline{\theta} - \frac{1-q}{p-q}c\right) & , \text{ for } \quad x \geq \underline{\theta} + \frac{1-\beta}{\beta(p-q)} \max\{c, (p-q)(w_H - w_L)\} \end{cases};$$

(b) If $(p-q)^2(\theta_H - \theta_L) < (1-q)c$, $B(x) = \beta x + (1-\beta)\underline{\theta}$, for $x \in [\underline{\theta}, +\infty)$.

Proof. Rewrite the optimization problem as follows,

$$\max_{e,r,U_L,U_H} \begin{bmatrix} e \\ 1-e \end{bmatrix}^\intercal \begin{bmatrix} p & 1-p \\ q & 1-q \end{bmatrix} \left(\begin{bmatrix} (1-\beta)\theta_H \\ (1-\beta)\theta_L \end{bmatrix} + \begin{bmatrix} r_H & 1-r_H \\ r_L & 1-r_L \end{bmatrix} \begin{bmatrix} \beta(U_H^A + U_H^P) \\ \beta(U_L^A + U_L^P) \end{bmatrix} \right) - e(1-\beta)c$$

subject to

$$e \in \arg\max_{e' \in [0,1]} U^A(e', r \mid w; U_L, U_H) \tag{39}$$

$$r \in \arg\max_{r' \in [0,1]^2} U^P(e, r' \mid w; U_L, U_H) \tag{40}$$

$$U_L, U_H \in \Lambda(x). \tag{41}$$

I proceed by considering two auxiliary optimization problems: one with the additional constraint $e = 0$ and one with the additional constraint $e > 0$. Therefore, $B(x)$ can be calculated by comparing the results from these two complementary optimization problems.

(1) $e = 0$.

It's straightforward to show $B_1(x) = (1-\beta)\underline{\theta} + \beta x$ for $U_L^A + U_L^P = U_H^A + U_H^P = x$;

(2) $e > 0$.

First notice $e > 0$ only if

$$U_H^P - U_L^P = \frac{1-\beta}{\beta}(w_H - w_L) \tag{42}$$

$$(r_H - r_L)(p-q)[(1-\beta)(w_H - w_L) + \beta(U_H^A - U_L^A)] \geq (1-\beta)c. \tag{43}$$

(42) holds because otherwise the principal reports the low/high output regardless of the true output so that the agent does not have incentives to exert effort. (43) is derived from (39).

Therefore, I proceed by assuming $x - \underline{\theta} \geq \frac{1-\beta}{\beta}(w_H - w_L)$ because, otherwise, (42) can not hold. Without loss of generality, I assume

$$U_L^P = \underline{\theta} - w_L \text{ and } U_H^P = \underline{\theta} - w_L + \frac{1-\beta}{\beta}(w_H - w_L) \tag{44}$$

$$\max\left\{U_L^A + U_L^P, U_H^A + U_H^P\right\} = x. \tag{45}$$

(i) If $U_H^A + U_H^P = x$, I have

$$U_H^A = x - \left(\underline{\theta} - w_L + \frac{1-\beta}{\beta}(w_H - w_L)\right) \text{ and } U_L^A \in [w_L, x - (\underline{\theta} - w_L)]$$

which implies $(1-\beta)(w_H - w_L) + \beta(U_H^A - U_L^A) \in [0, \beta(x - \underline{\theta})]$. Therefore, $r_H > r_L$ according to (43) which implies

$$U_H^A - U_L^A \geq \frac{1-\beta}{\beta(p-q)(r_H - r_L)}c - \frac{1-\beta}{\beta}(w_H - w_L) \geq \frac{1-\beta}{\beta(p-q)}c - \frac{1-\beta}{\beta}(w_H - w_L).$$

The first equality always holds since otherwise U_L^A can be increased without changing e, r_L and r_H resulting in a greater value of the objective function. So I can rewrite the optimization problem as follows,

$$\max_{e, r_L, r_H \in [0,1]} \begin{bmatrix} e \\ 1-e \end{bmatrix}^{\intercal} \begin{bmatrix} p & 1-p \\ q & 1-q \end{bmatrix} \left(\begin{bmatrix} (1-\beta)\theta_H \\ (1-\beta)\theta_L \end{bmatrix} + \begin{bmatrix} r_H & 1-r_H \\ r_L & 1-r_L \end{bmatrix} \begin{bmatrix} \beta x \\ \beta y \end{bmatrix} \right) - e(1-\beta)c$$

subject to

$$\frac{1-\beta}{\beta(p-q)(r_H - r_L)}c \in [0, x - \underline{\theta}]$$

where

$$y = x - \frac{1-\beta}{\beta(p-q)(r_H - r_L)}c.$$

The constraint set is non-empty if and only if $x \geq \underline{\theta} + \frac{1-\beta}{\beta(p-q)}c$. As a result, the maximum value is $(1-\beta)(\bar{\theta} - \frac{1-q}{p-q}c) + \beta x$ for $e = r_H = 1$ and $r_L = 0$.

(ii) If $U_L^A + U_L^P = x$, I have

$$U_H^A \in \left[w_L, x - \left(\underline{\theta} - w_L + \frac{1-\beta}{\beta}(w_H - w_L)\right)\right] \text{ and } U_L^A = x - (\underline{\theta} - w_L)$$

which implies $(1-\beta)(w_H - w_L) + \beta(U_H^A - U_L^A) \in [(1-\beta)(w_H - w_L) - \beta(x - \underline{\theta}), 0]$. Therefore, $\rho_H < \rho_L$ according to (43) which implies

$$U_H^A - U_L^A \leq \frac{1-\beta}{\beta(p-q)(r_H - r_L)}c - \frac{1-\beta}{\beta}(w_H - w_L) \leq -\frac{1-\beta}{\beta(p-q)}c - \frac{1-\beta}{\beta}(w_H - w_L).$$

The first equality always holds since otherwise U_H^A can be increased without changing e, r_L and r_H resulting in a greater value of the objective function. So I can rewrite the optimization problem as follows,

$$\max_{e, r_L, r_H \in [0,1]} \begin{bmatrix} e \\ 1-e \end{bmatrix}^{\intercal} \begin{bmatrix} p & 1-p \\ q & 1-q \end{bmatrix} \left(\begin{bmatrix} (1-\beta)\theta_H \\ (1-\beta)\theta_L \end{bmatrix} + \begin{bmatrix} r_H & 1-r_H \\ r_L & 1-r_L \end{bmatrix} \begin{bmatrix} \beta y \\ \beta x \end{bmatrix} \right) - e(1-\beta)c$$

subject to

$$\frac{1-\beta}{\beta(p-q)(r_H - r_L)}c \in \left[\frac{1-\beta}{\beta}(w_H - w_L) - (x - \underline{\theta}), 0\right]$$

where

$$y = x + \frac{1-\beta}{\beta(p-q)(r_H - r_L)}c.$$

The constraint set is non-empty if and only if $x \geq \underline{\theta} + \frac{1-\beta}{\beta(p-q)}c + \frac{1-\beta}{\beta}(w_H - w_L)$. As a result, the maximum value is $(1-\beta)\left(\overline{\theta} - \frac{1-q}{p-q}c\right) + \beta x$ for $e = r_L = 1$ and $r_H = 0$.

Hence, I conclude $B_2(x) = (1-\beta)\left(\overline{\theta} - \frac{1-q}{p-q}c\right) + \beta x$ if $x \geq \underline{\theta} + \frac{1-\beta}{\beta(p-q)}c$.

Therefore,

$$B(x) = \begin{cases} \max\{B_1(x), B_2(x)\}, & \text{if } x \geq \underline{\theta} + \frac{1-\beta}{\beta(p-q)}c \\ B_1(x) & \text{, if otherwise} \end{cases}.$$

The result follows by algebra. ■

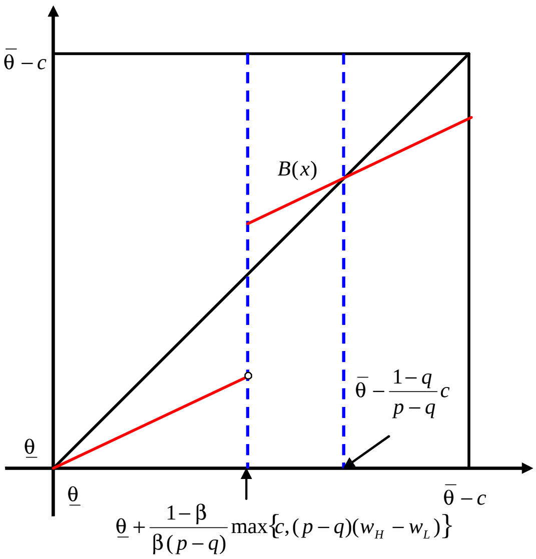

Figure 1.11

The function $B(x)$ is illustrated in the figure above. Therefore, as long as $B(x) \geq x$ for $x = \underline{\theta} + \frac{1-\beta}{\beta(p-q)} \max\{c, (p-q)(w_H - w_L)\}$, $B^\infty(\bar{\theta} - c) = \bar{\theta} - \frac{1-q}{p-q}c$. Otherwise, $B^\infty(\bar{\theta} - c) = \underline{\theta}$.

This completes the proof.

1.5.4 Proof of Theorem 1.1

Let $U^i(j)$ for $i \in \{A, P\}$ and $j \in \{C, D\}$ denote the player i's NED payoff at state j. Apparently, I have $U^A(D) = w_L^*$ and $U^P(D) = \underline{\theta} - w_L^*$. Furthermore,

$$U^A(C) = (1-\beta)(\overline{w} - c) + \beta[pU^A(C) + (1-p)((1-r_L)U^A(D) + r_L U^A(C))]$$

$$U^P(C) = (1-\beta)(\bar{\theta} - \overline{w}) + \beta[pU^P(C) + (1-p)((1-r_L)U^P(D) + r_L U^P(C))]$$

where $\overline{w} = pw_H^* + (1-p)[(1-r_L)w_L^* + r_L w_H^*]$. By algebra, I have

$$U^A(C) = \frac{(1-\beta)(\overline{w} - c) + \beta(1-p)(1-r_L)w_L}{1 - \beta p - \beta(1-p)r_L}$$

$$U^P(C) = \frac{(1-\beta)(\bar{\theta} - \overline{w}) + \beta(1-p)(1-r_L)(\underline{\theta} - w_L)}{1 - \beta p - \beta(1-p)r_L}$$

so that

$$U^A(C) + U^P(C) = \frac{(1-\beta)(\bar{\theta} - c) + \beta(1-p)(1-r_L)\underline{\theta}}{1 - \beta p - \beta(1-p)r_L} = \bar{\theta} - \frac{1-q}{p-q}c$$

as claimed.

The incentive compatibility at state D is straightforward. At state C, given the principal's strategy of reporting truthfully, the agent's NED payoff of shirking and exerting effort are

$$(1-\beta)\underline{w}+\beta[qU^A(C)+(1-q)((1-r_L)U^A(D)+r_LU^A(C))]$$

$$(1-\beta)(\overline{w}-c)+\beta[pU^A(C)+(1-p)((1-r_L)U^A(D)+r_LU^A(C))]$$

respectively which are equal by algebra where $\underline{w}=qw_H^*+(1-q)[(1-r_L)w_L^*+r_Lw_H^*]$. So it's optimal for the agent to exert effort at State C. Also in state C, given the agent's strategy of exerting effort, the expected utilities for the principal to report the low output and the high output (regardless of the true output) are

$$-(1-\beta)w_L^*+\beta U^P(D) \text{ and } -(1-\beta)w_H^*+\beta U^P(C)$$

respectively which are equal by algebra. So it's optimal for the principal to report truthfully at state C. And $\beta \in \left[\frac{c}{(p-q)^2(\theta_H-\theta_L)+qc},1\right)$ guarantees $r_L\in[0,1]$.

This completes the proof.

1.5.5 Proof of Proposition 1.4

For $x\in[\underline{\theta},+\infty)$, $\Lambda_1(x)\equiv\{(U^A,U^P)\mid \underline{\theta}\le U^A+U^P\le x\}$. And define a mapping $\Pi_1: 2^{\mathbb{R}^2}\to 2^{\mathbb{R}^2}$ as

$$\Pi_1(X)=\left\{(U^A,U^P)\;\middle|\;\begin{array}{c}\exists U_L,U_H\in co(X),\ w\in\mathbb{R}^2,\ e\in[0,1] \text{ and } r\in[0,1]^2\\ \text{such that}\\ U^A=U^A((e,r)\mid w;U_L,U_H)\\ U^P=U^P((e,r)\mid w;U_L,U_H)\\ e\in\arg\max_{e'\in[0,1]}U^A((e',r)\mid w;U_L,U_H)\\ r\in\arg\max_{r'\in[0,1]^2}U^P((e,r')\mid w;U_L,U_H)\end{array}\right\}.$$

Now, the set of the feasible payoff pairs V is defined as

$$V=\{(U^A,U^P)\mid \underline{\theta}\le U^A+U^P\le\overline{\theta}-c\}$$

because the payoffs are transferable between the agent and the principal. Therefore, there exists $x\in[\underline{\theta},\overline{\theta}-c]$ such that

$$co(\Pi_1^\infty(V))=\{(U^A,U^P)\mid \underline{\theta}\le U^A+U^P\le x\}$$

which implies that I can iterate on x instead of the whole set. By applying the procedure in the proof of Proposition 1.3, I have

$$x = \begin{cases} \underline{\theta} & , \text{for } \beta \in \left(0, \frac{c}{(p-q)^2(\theta_H-\theta_L)+qc}\right) \\ \overline{\theta} - \frac{1-q}{p-q}c, & \text{for } \beta \in \left[\frac{c}{(p-q)^2(\theta_H-\theta_L)+qc}, 1\right) \end{cases}.$$

That completes the proof.

1.5.6 Proof of Lemma 1.1

The case of $w_H = 0$ is trivial so that I proceed by assuming $w_H > 0$.

Notice, given any history (ϑ^t, e^t), the agent believes that the history of the true outputs θ^t is consistent with the history of the reported outputs ϑ^t due to $p, q \in (0,1)$ which implies the principal's potential deviations are undetectable. That in turn implies that the agent believes of the principal being going to report truthfully in the future. Therefore, the agent's strictly dominant strategy is shirking if $w_H < \frac{c}{p-q}$ and exerting effort if $w_H > \frac{c}{p-q}$. In either case, the principal does not have incentives to report the high output as long as $w_H > 0$ because the agent's continuation strategy is independent of the principal's report. A contradiction.

1.5.7 Proof of Theorem 1.2

By definition, there exists a PPE (e, r) with respect to w^{**} $\forall \mathbf{e} \in \mathbb{C}$ such that r features truth-reporting and

$$\mathbf{e} = (1-\beta)e + \beta \begin{bmatrix} e(1-p) + (1-e)(1-q) \\ ep + (1-e)q \end{bmatrix}^T \begin{bmatrix} \mathbf{e}_L \\ \mathbf{e}_H \end{bmatrix} \tag{46}$$

where

$$\mathbf{e}_\vartheta = (1-\beta)\mathbb{E}_1 \left[\sum_{t=1}^{\infty} \beta^{t-1} e_t \middle| s; \{\vartheta_0 = \vartheta\} \right] \in \mathbb{C} \ \forall \vartheta \in \Theta.$$

Regardless of the true output, the principal's NED payoff following the reported low output and the reported high output are

$$\beta[\mathbf{e}_L(\overline{\theta} - pw_H^{**}) + (1-\mathbf{e}_L)(\underline{\theta} - qw_H^{**})]$$

$$-(1-\beta)w_H^{**} + \beta[\mathbf{e}_H(\overline{\theta} - pw_H^{**}) + (1-\mathbf{e}_H)(\underline{\theta} - qw_H^{**})]$$

respectively which imply

$$\mathbf{e}_H - \mathbf{e}_L = \frac{1-\beta}{\beta} \frac{c}{(p-q)[(p-q)(\theta_H - \theta_L) - c]} \equiv y > 0 \tag{47}$$

because, otherwise, the principal strictly prefers reporting the low output or reporting the high output.

By substituting $\mathbf{e}_H$ in (46) by (47), I have

$$\mathbf{e} = (1-\beta)e + \beta\mathbf{e}_L + \beta[ep + (1-e)q]y$$

which implies that given $\mathbf{e}$, the maximum $\mathbf{e}_L$ denoted by $\overline{\mathbf{e}}_L$ is achieved by $e = 0$. Therefore,

$$\overline{\mathbf{e}}_L = \frac{1}{\beta}\mathbf{e} - qy$$

as illustrated in the following figure,

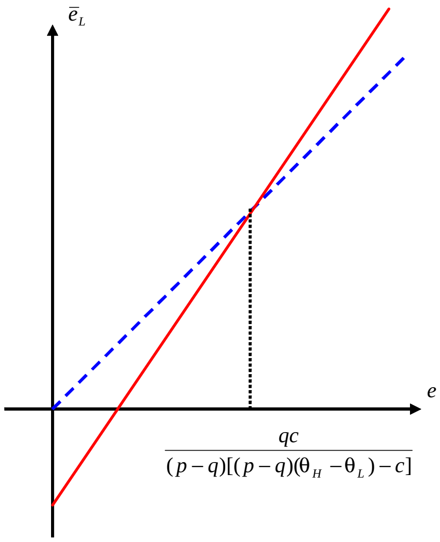

Figure 1.12

Hence, I conclude $\mathbf{e} \geq \frac{qc}{(p-q)[(p-q)(\theta_H-\theta_L)-c]}$. Otherwise, $\mathbf{e}_L$ drops below zero eventually.

By substituting $\mathbf{e}_L$ in (46) by (47), I have

$$\mathbf{e} = (1-\beta)e + \beta\mathbf{e}_H - \beta[e(1-p) + (1-e)(1-q)]y$$

which implies that given $\mathbf{e}$, the minimum $\mathbf{e}_H$ denoted by $\underline{\mathbf{e}}_H$ is achieved by $e = 1$. Therefore,

$$\underline{\mathbf{e}}_H = \frac{1}{\beta}\mathbf{e} + (1-p)y - \frac{1-\beta}{\beta}$$

which implies $\mathbf{e} \leq \frac{(p-q)^2(\theta_H-\theta_L)-(1-q)c}{(p-q)[(p-q)(\theta_H-\theta_L)-c]}$. Otherwise, $\mathbf{e}_H$ rises above one eventually.

Therefore,

$$\mathbb{C} \subseteq \left[\frac{qc}{(p-q)[(p-q)(\theta_H-\theta_L)-c]}, \frac{(p-q)^2(\theta_H-\theta_L)-(1-q)c}{(p-q)[(p-q)(\theta_H-\theta_L)-c]}\right] \equiv [\underline{\mathbf{e}}, \overline{\mathbf{e}}]$$

if $(p-q)^2(\theta_H-\theta_L) \geq c$. Otherwise, $\mathbb{C}$ is empty. Notice, if $\beta < \frac{c}{(p-q)^2(\theta_H-\theta_L)}$, the interval above is not large enough to make (47) hold. So I proceed by assuming otherwise.

In order to complete the proof, it suffices to show there exists a PPE with respect to w^{**} $\forall \mathbf{e} \in [\underline{\mathbf{e}}, \overline{\mathbf{e}}]$ such that r features truth-reporting and (46). I solve for $\mathbf{e} = \underline{\mathbf{e}}, \overline{\mathbf{e}}$ while leaving the rest for the readers as follows: for $\mathbf{e} = \underline{\mathbf{e}}$, $e_t = 0$ if $t = 1$ or $\vartheta_{t-1} = L$ and $e_t = \frac{1}{\beta}\frac{c}{(p-q)^2(\theta_H-\theta_L)}$ otherwise; for $\mathbf{e} = \overline{\mathbf{e}}$, $e_t = 1$ if $t = 1$ or $\vartheta_{t-1} = H$ and $e_t = 1 - \frac{1}{\beta}\frac{c}{(p-q)^2(\theta_H-\theta_L)}$ otherwise.

1.5.8 Proof of Lemma 1.2

According to Proposition 12.2.2 by Mailath and Samuelson (2006), it suffices to prove (20) and (21) are no profitable one-shot deviations conditions. Notice in general, this game does not satisfy full support assumption which is required by Proposition 12.2.2. Fortunately, it will not be a problem for the case under consideration in which the principal reports truthfully so that (3) is well-defined everywhere.

If the agent is at state $\gamma \in \Gamma$, then the principal's NED payoff from following the equilibrium truth-reporting strategy is

$$\Bbbk(\gamma) \equiv (1-\mathbf{e}_\gamma)\left[(1-q)\theta_L + q\left(\theta_H - \frac{c}{p-q}\right)\right] + \mathbf{e}_\gamma\left[(1-p)\theta_L + p\left(\theta_H - \frac{c}{p-q}\right)\right].$$

Given $\lambda \in \Lambda$ and $x \in \mathbb{N}(\lambda)$, if the true output is low, then the principal assigns the probability $F(x, L; \lambda)$ to the agent having exerted effort. Hence, the principal's NED payoff from reporting the high output instead in current period but reporting truthfully thereafter is

$$(1-\beta)\left(-\frac{c}{p-q}\right) + \beta[(1-F(x,L;\lambda))\Bbbk(H,0) + F(x,L;\lambda)\Bbbk(H,1)]$$

which is supposed to be lower than the principal's NED payoff from following the equilibrium truth-reporting strategy

$$\beta[(1-F(x,L;\lambda))\Bbbk(L,0) + F(x,L;\lambda)\Bbbk(L,1)].$$

This gives (20). A similar argument gives (21).

1.5.9 Proof of Proposition 1.5

By algebra, I conclude the belief-updating function $F : [0,1] \times \Theta \times \Lambda \to [0,1]$ has the following properties:

(a) $0 < F(x, L; \lambda) < \overline{e}(x; \lambda) < F(x, H; \lambda) < 1$ for $\overline{e}(x; \lambda) \in (0, 1)$;

(b) $F(x, L; \lambda) = \overline{e}(x; \lambda) = F(x, H; \lambda)$ for $\overline{e}(x; \lambda) = 0$ or 1;

(c) $F(x, L; \lambda)$ and $F(x, H; \lambda)$ are monotone and continuous in x;

(d) $F(x, L; \lambda)$ is convex in x and $F(x, H; \lambda)$ is concave in x.

$\forall \lambda \in \Lambda$ and $x \in \mathbb{N}(\lambda)$,

$$(F(x, H; \lambda) - F(x, L; \lambda))[(\mathbf{e}_{H,1} - \mathbf{e}_{L,1}) - (\mathbf{e}_{H,0} - \mathbf{e}_{L,0})] \geq 0$$

following (20) and (21). Remember $F(x, H; \lambda) \geq F(x, L; \lambda)$ $\forall \lambda \in \Lambda$ and $x \in \mathbb{N}(\lambda)$ by (a) and (b).

Suppose there exist λ and $x \in \mathbb{N}(\lambda)$ such that $F(x, H; \lambda) > F(x, L; \lambda)$. Then $\mathbf{e}_{H,0} - \mathbf{e}_{L,0} \leq \mathbf{e}_{H,1} - \mathbf{e}_{L,1}$. Furthermore, if $y < \mathbf{e}_{H,0} - \mathbf{e}_{L,0}$, (20) does not hold. If $y > \mathbf{e}_{H,1} - \mathbf{e}_{L,1}$, (21) does not hold. So the result follows.

Suppose $F(x, H; \lambda) = F(x, L; \lambda)$ $\forall \lambda \in \Lambda$ and $x \in \mathbb{N}(\lambda)$. This implies $\overline{e}(x; \lambda) = 0$ or 1 $\forall \lambda \in \Lambda$ and $x \in \mathbb{N}(\lambda)$ by (a) and (b). If there exist $\lambda \in \Lambda$ and $x \in \mathbb{N}(\lambda)$ such that $\overline{e}(x; \lambda) = 0$, then $F(x, L; \lambda) = F(x, H; \lambda) = 0$ which implies $\mathbf{e}_{H,0} - \mathbf{e}_{L,0} = y$ by (20) and (21). If there exist $\lambda \in \Lambda$ and $x \in \mathbb{N}(\lambda)$ such that $\overline{e}(x; \lambda) = 1$, then $F(x, L; \lambda) = F(x, H; \lambda) = 1$ which implies $\mathbf{e}_{H,1} - \mathbf{e}_{L,1} = y$ by (20) and (21). Therefore, $\mathbf{e}_{H,0} - \mathbf{e}_{L,0} = y = \mathbf{e}_{H,1} - \mathbf{e}_{L,1}$ if $\overline{e}(x; \lambda) = 0$ for some $\lambda \in \Lambda$ and $x \in \mathbb{N}(\lambda)$ and $\overline{e}(x; \lambda) = 1$ for some $\lambda \in \Lambda$ and $x \in \mathbb{N}(\lambda)$. This leaves us two cases: (1) suppose $\overline{e}(x; \lambda) = 0$ $\forall \lambda \in \Lambda$ and $x \in \mathbb{N}(\lambda)$. If $\overline{e}(x; \lambda) = 0$ for $\lambda \in \Lambda$ and $x \in \mathbb{N}_t(\lambda)$, then $F(x, \theta; \lambda) = 0 \in \mathbb{N}_{t+1}(L)$ and $\mathbb{N}_{t+1}(H)$ by (17). Hence, $\overline{e}(F(x, \theta; \lambda); L) = \overline{e}(F(x, \theta; \lambda); H) = 0$ which implies $e_{L,0} = e_{H,0} = 0$, and in turn $\mathbf{e}_{H,0} - \mathbf{e}_{L,0} = 0$. This contradicts (20) and (21) given $y > 0$; (2) suppose $\overline{e}(x; \lambda) = 1$ $\forall \lambda \in \Lambda$ and $x \in \mathbb{N}(\lambda)$. A similar contradiction is reached.

This completes the proof.

1.5.10 Proof of Proposition 1.6

Following Ely, Hörner and Olszewski (2005), I say the effort function e supporting truth-reporting is belief-free if the principal has appropriate incentive to report truthfully even when she believes with certainty of the agent having either shirked or exerted effort. Suppose the principal believes of the agent having shirked. She reports truthfully if and only if $\mathbf{e}_{H,0} - \mathbf{e}_{L,0} = y$ due to (20) and (21). Suppose the principal believes of the agent having exerted effort. She reports truthfully if and only if $\mathbf{e}_{H,1} - \mathbf{e}_{L,1} = y$ due to (20) and (21). So I conclude the effort function e supporting truth-reporting is belief-free if and only if $\mathbf{e}_{H,0} - \mathbf{e}_{L,0} = y = \mathbf{e}_{H,1} - \mathbf{e}_{L,1}$. Hence,

$$e^* \in \underset{e_{L,0},e_{L,1},e_{H,0},e_{H,1}}{\arg\max} \left\{ \frac{e_0}{1-(e_1-e_0)} \right\}$$

subject to

$$\frac{1-\beta}{1-\beta(e_1-e_0)}(e_{H,0}-e_{L,0}) = y \tag{48}$$

$$\frac{1-\beta}{1-\beta(e_1-e_0)}(e_{H,1}-e_{L,1}) = y \tag{49}$$

$$e_{L,0}, e_{L,1}, e_{H,0}, e_{H,1} \in [0,1]. \tag{50}$$

By (48) and (49), I have $e^*_{H,0} - e^*_{L,0} = e^*_{H,1} - e^*_{L,1} > 0$. Notice either $e^*_{H,0}$ or $e^*_{H,1}$ must be one. Otherwise, e' with $e'_\gamma = e_\gamma + \min\{1-e^*_{H,0}, 1-e^*_{H,1}\}$ $\forall \gamma \in \Gamma$ satisfies (48)-(50) and is strictly more efficient than e^*. A contradiction.

Case 1: Suppose $e^*_{H,0} = 1$.

Assume $e^*_{H,1} < 1$ which implies there exists $\varepsilon > 0$ such that $e' = (e^*_{L,0}\ e^*_{L,1}+\varepsilon\ e^*_{H,0}\ e^*_{H,1}+\varepsilon)^T$ satisfy (50) with

$$\frac{1-\beta}{1-\beta(e'_1-e'_0)}(e'_{H,0}-e'_{L,0}) = \frac{1-\beta}{1-\beta(e'_1-e'_0)}(e'_{H,1}-e'_{L,1}) > y.$$

Furthermore, there exists $\eta > 0$ such that $e'' = (e'_{L,0}+\eta\ e'_{L,1}+\eta\ e'_{H,0}\ e'_{H,1})^T$ satisfies (48)-(50) which is strictly more efficient than e^*. A contradiction. So I conclude $e^*_{H,1} = 1$ and the result follows.

Case 2: Suppose $e^*_{H,1} = 1$.

Let $z \equiv e_{H,0} - e_{L,0}$ so that I can rewrite the optimization problem as follows,

$$\{e^*_{H,0}, e^*_{H,0} - e^*_{L,0}\} \in \underset{e_{H,0},t}{\arg\max} \left\{ \frac{e_{H,0} - (1-q)z}{1-[(p-q)z + (1-e_{H,0})]} \right\}$$

subject to

$$\frac{1-\beta}{1-\beta[(p-q)z+(1-e_{H,0})]} z = y \tag{51}$$

$$e_{H,0}, e_{H,0} - z \in [0,1]. \tag{52}$$

From (51), I have

$$(p-q)z + (1-e_{H,0}) = \frac{1}{\beta} - \frac{(1-\beta)z}{\beta y} \text{ with } \frac{de_{H,0}}{dz} = \frac{(p-q)^2(\theta_H-\theta_L)}{c} \geq 1.$$

Inserting it into the objective function gives

$$\frac{1-(1-p)z-\left[\frac{1}{\beta}-\frac{(1-\beta)z}{\beta y}\right]}{1-\left[\frac{1}{\beta}-\frac{(1-\beta)z}{\beta y}\right]}$$

which is increasing in z. Hence, it is optimal to have $e^*_{H,0}=1$ and $z^*=\frac{1}{\beta}\frac{c}{(p-q)^2(\theta_H-\theta_L)}$. The result follows immediately.

This completes the proof.

1.5.11 Proof of Proposition 1.7

Since $e^*_L < e^*_H = 1$, I have

$$\frac{e^*_L p}{(1-e^*_L)q+e^*_L p} < 1 = \frac{e^*_H(1-p)}{(1-e^*_H)(1-q)+e^*_H(1-p)}$$

which implies that there exists $\varepsilon > 0$ such that for any decision function $e \in B_\varepsilon(e^*)$,

$$\begin{aligned} F(x,H;L) &\leq \frac{(e^*_L+\varepsilon)p}{[1-(e^*_L+\varepsilon)]q+(e^*_L+\varepsilon)p} \\ &< \frac{(e^*_H-\varepsilon)(1-p)}{[1-(e^*_H-\varepsilon)](1-q)+(e^*_H-\varepsilon)(1-p)} \leq F(x,L;H) \end{aligned}$$

$\forall x \in [0,1]$. Therefore, for any $e \in B_\varepsilon(e^*)$, there does not exist $\mathbf{F}$ such that $\mathbb{Q}(L)$ and $\mathbb{Q}(H)$ can be separated. The result follows.

1.5.12 Proof of Proposition 1.8

I first show $\mathbb{N}_t(L) \subseteq \mathbb{N}_t(H)$ $\forall t \geq 1$. Suppose $x \in \mathbb{N}_t(L)$ which implies that $\exists \lambda' \in \Lambda$ and $(L,\theta) \in \Theta^2$ such that $F(x',\theta;\lambda')$ for some $x' \in \mathbb{N}_t(\lambda')$. So for $\lambda' \in \Lambda$ and $(H,\theta) \in \Theta^2$, I have $\varsigma(\lambda',H,\theta) = H$ and $x = F(x',\theta;\lambda')$ for $x' \in \mathbb{N}_t(\lambda')$ which implies $x \in \mathbb{N}_t(H)$. Therefore, $\mathbb{N}_t(L) \subseteq \mathbb{N}_t(H)$. A similar argument establishes $\mathbb{N}_t(H) \subseteq \mathbb{N}_t(L)$. The result follows.

Therefore, it suffices to prove $\inf \mathbb{N}_1 \leq \frac{\boldsymbol{\rho}_{\lambda,1}}{\boldsymbol{\rho}_{\lambda,0}+\boldsymbol{\rho}_{\lambda,1}} \leq \sup \mathbb{N}_1$ $\forall \lambda \in \Gamma$. By (**??**), I have

$$\boldsymbol{\rho}_{L,0} = (1-q)\sum\nolimits_{\gamma\in\Gamma}(1-e_\gamma)\boldsymbol{\rho}_\gamma \text{ and } \boldsymbol{\rho}_{L,1} = (1-p)\sum\nolimits_{\gamma\in\Gamma} e_\gamma \boldsymbol{\rho}_\gamma$$

and in turn

$$\frac{\boldsymbol{\rho}_{L,1}}{\boldsymbol{\rho}_{L,0}+\boldsymbol{\rho}_{L,1}} = \frac{(1-p)\sum_{\gamma\in\Gamma} e_\gamma \boldsymbol{\rho}_\gamma}{(1-q)\sum_{\gamma\in\Gamma}(1-e_\gamma)\boldsymbol{\rho}_\gamma+(1-p)\sum_{\gamma\in\Gamma} e_\gamma \boldsymbol{\rho}_\gamma}$$

which implies $\frac{\boldsymbol{\rho}_{L,1}}{\boldsymbol{\rho}_{L,0}+\boldsymbol{\rho}_{L,1}}$ goes between

$$\frac{(1-p)\sum_{\gamma\in\{(L,0),(L,1)\}} e_\gamma \boldsymbol{\rho}_\gamma}{(1-q)\sum_{\gamma\in\{(L,0),(L,1)\}}(1-e_\gamma)\boldsymbol{\rho}_\gamma + (1-p)\sum_{\gamma\in\{(L,0),(L,1)\}} e_\gamma \boldsymbol{\rho}_\gamma} = F\left(\frac{\boldsymbol{\rho}_{L,1}}{\boldsymbol{\rho}_{L,0}+\boldsymbol{\rho}_{L,1}}, L; L\right)$$

and

$$\begin{aligned}&\frac{(1-p)\sum_{\gamma\in\{(H,0),(H,1)\}} e_\gamma \boldsymbol{\rho}_\gamma}{(1-q)\sum_{\gamma\in\{(H,0),(H,1)\}}(1-e_\gamma)\boldsymbol{\rho}_\gamma + (1-p)\sum_{\gamma\in\{(H,0),(H,1)\}} e_\gamma \boldsymbol{\rho}_\gamma}\\ &= F\left(\frac{\boldsymbol{\rho}_{H,1}}{\boldsymbol{\rho}_{H,0}+\boldsymbol{\rho}_{H,1}}, L; H\right).\end{aligned}$$

A similar argument applies for $\frac{\boldsymbol{\rho}_{H,1}}{\boldsymbol{\rho}_{H,0}+\boldsymbol{\rho}_{H,1}}$.

(b) (Sufficiency) Suppose (20) and (21) are satisfied $\forall\lambda\in\Lambda$ and $x\in\{\underline{x},\overline{x}\}$. Then, $\forall\lambda\in\Lambda$ and $x = t\underline{x} + (1-t)\overline{x}$ for some $t\in[0,1]$,

$$\begin{aligned}&(1-F(x,L;\lambda))(\mathbf{e}_{H,0}-\mathbf{e}_{L,0}) + F(x,L;\lambda)(\mathbf{e}_{H,1}-\mathbf{e}_{L,1})\\ &= (\mathbf{e}_{H,0}-\mathbf{e}_{L,0}) + F(x,L;\lambda)[(\mathbf{e}_{H,1}-\mathbf{e}_{L,1}) - (\mathbf{e}_{H,0}-\mathbf{e}_{L,0})]\\ &\leq (\mathbf{e}_{H,0}-\mathbf{e}_{L,0}) + [tF(\underline{x},L;\lambda) + (1-t)F(\overline{x},L;\lambda)][(\mathbf{e}_{H,1}-\mathbf{e}_{L,1}) - (\mathbf{e}_{H,0}-\mathbf{e}_{L,0})]\\ &\leq y\end{aligned}$$

where the first inequality follows the facts (1) $\mathbf{e}_{H,0}-\mathbf{e}_{L,0} \leq \mathbf{e}_{H,1}-\mathbf{e}_{L,1}$; (2) $F(x,L;\lambda)$ is convex in x. And

$$\begin{aligned}&(1-F(x,H;\lambda))(\mathbf{e}_{H,0}-\mathbf{e}_{L,0}) + F(x,H;\lambda)(\mathbf{e}_{H,1}-\mathbf{e}_{L,1})\\ &= (\mathbf{e}_{H,0}-\mathbf{e}_{L,0}) + F(x,H;\lambda)[(\mathbf{e}_{H,1}-\mathbf{e}_{L,1}) - (\mathbf{e}_{H,0}-\mathbf{e}_{L,0})]\\ &\geq (\mathbf{e}_{H,0}-\mathbf{e}_{L,0}) + [tF(\underline{x},H;\lambda) + (1-t)F(\overline{x},H;\lambda)][(\mathbf{e}_{H,1}-\mathbf{e}_{L,1}) - (\mathbf{e}_{H,0}-\mathbf{e}_{L,0})]\\ &\geq y\end{aligned}$$

where the first inequality follows the facts (1) $\mathbf{e}_{H,0}-\mathbf{e}_{L,0} \leq \mathbf{e}_{H,1}-\mathbf{e}_{L,1}$ by Lemma 9; (2) $F(x,H;\lambda)$ is concave in x.

(Necessarily) By definition of $\underline{x}$, there exists a sequence $\left\{x_n \in \bigcup_{t=1}^{\infty}\mathbb{N}_t\right\}_{n=1}^{\infty}$ such that $\underline{x} = \lim_{n\to\infty} x_n$ and x_n satisfies (20) and (21). Therefore, by taking n to ∞, I conclude $\underline{x}$ satisfies (20) and (21) as well since $F(x,\theta;\lambda)$ is continuous in x. A similar argument applies for $\overline{x}$.

This completes the proof.

1.5.13 Proof of Theorem 1.3

(a) The existence of $\varepsilon > 0$ such that either $e_{H,0} < 1 - \varepsilon$ or $e_{H,1} < 1 - \varepsilon$ has been proven in the paper.

(b) Suppose $\underline{x} = 0$. Since $\underline{x} = \inf \bigcup_{t=1}^{\infty} \mathbb{N}_t$, there exists a sequence $\left\{x_n \in \bigcup_{t=1}^{\infty} \mathbb{N}_t\right\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} x_n = 0$. By (17), there exist $\theta_n \in \Theta$, $\lambda_n \in \Lambda$ and $x'_n \in \mathbb{N}_{t_n}(\lambda_n)$ for some $t_n \geq 0$ such that $x_n = F(x'_n, \theta_n; \lambda_n)$. Then $\lim_{n\to\infty} F(x'_n, \theta_n; \lambda_n) = 0$ which implies $\mathbf{e}_{H,0} - \mathbf{e}_{L,0} = y$ by (20) and (21). Since the effort function is belief-based, I have $\mathbf{e}_{H,1} - \mathbf{e}_{L,1} \neq y$ which implies $F(x, \theta; \lambda) = 0\ \forall \theta \in \Theta$, $\lambda \in \Lambda$ and $x \in \mathbb{N}(\lambda)$ by (20) and (21), or equivalently, $\mathbf{F} = 0$ by (26). A contradiction with $\underline{\mathbf{F}} > 0$.

Suppose $\overline{x} = 1$. Since $\overline{x} = \sup \bigcup_{t=1}^{\infty} \mathbb{N}_t$, there exists a sequence $\left\{x_n \in \bigcup_{t=1}^{\infty} \mathbb{N}_t\right\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} x_n = 1$. By (17), there exist $\theta_n \in \Theta$, $\lambda_n \in \Lambda$ and $x'_n \in \mathbb{N}_{t_n}(\lambda_n)$ for some $t_n \geq 0$ such that $x_n = F(x'_n, \theta_n; \lambda_n)$. Then $\lim_{n\to\infty} F(x'_n, \theta_n; \lambda_n) = 1$ which implies $\mathbf{e}_{H,1} - \mathbf{e}_{L,1} = y$ by (20) and (21). Since the effort function is belief-based, I have $\mathbf{e}_{H,0} - \mathbf{e}_{L,0} \neq y$ which implies $F(x, \theta; \lambda) = 1\ \forall \theta \in \Theta$, $\lambda \in \Lambda$ and $x \in \mathbb{N}(\lambda)$ by (20) and (21), or equivalently, $\mathbf{F} = 1$ by (26). A contradiction with $\overline{\mathbf{F}} < 1$.

Since $\mathbf{F} \in [\underline{\mathbf{F}}, \overline{\mathbf{F}}]$, I conclude $\eta < \underline{x} \leq \overline{x} < 1 - \eta$ for some $\eta > 0$. I will come back to prove the second inequality holds strictly in the end.

Suppose $e_{L,0} = 0$. If $e_1 = 1$, then $\frac{\rho_{L,1}}{\rho_{L,0}+\rho_{L,1}} = \frac{\rho_{H,1}}{\rho_{H,0}+\rho_{H,1}} = 1$ which implies $\overline{x} = 1$. A contradiction. So I proceed by assuming $e_1 < 1$ which implies $\frac{\rho_{L,1}}{\rho_{L,0}+\rho_{L,1}}, \frac{\rho_{H,1}}{\rho_{H,0}+\rho_{H,1}} < 1$. Consider the principal has the initial state L followed by the history $\{(L, L), (L, L), \cdots\}$ which implies the principal always observes the true output to be low then reports truthfully. Define $x_0 = \frac{\rho_{L,1}}{\rho_{L,0}+\rho_{L,1}} < 1$ and $x_{t+1} = F(x_t, L; L)$. Hence, $\{x_t\}_{t=0}^{\infty}$ is a decreasing sequence because $x_{t+1} = F(x_t, L; L) \leq \overline{e}(x_t; L) \leq x_t$ where the second inequality follows the fact $e_{L,0} = 0$. Without loss of generality, assume $e_{L,1} > 0$ which implies $0 < x_{t+1} < x_t$ if $x_t > 0$. Therefore, $\lim_{t\to\infty} x_t = 0$ which implies $\underline{x} = 0$. A contradiction.

Suppose $e_{H,1} = 1$. If $e_0 = 0$, then $\frac{\rho_{L,1}}{\rho_{L,0}+\rho_{L,1}} = \frac{\rho_{H,1}}{\rho_{H,0}+\rho_{H,1}} = 0$ which implies $\underline{x} = 0$. A contradiction. So I proceed by assuming $e_0 > 0$ which implies $\frac{\rho_{L,1}}{\rho_{L,0}+\rho_{L,1}}, \frac{\rho_{H,1}}{\rho_{H,0}+\rho_{H,1}} > 0$. Consider the principal has the initial state H followed by the history $\{(H, H), (H, H), \cdots\}$ which implies the principal always observes the true output to be high then reports truthfully. Define $x_0 = \frac{\rho_{H,1}}{\rho_{H,0}+\rho_{H,1}} > 0$ and $x_{t+1} = F(x_t, H; H)$. Hence, $\{x_t\}_{t=0}^{\infty}$ is an increasing sequence because $x_{t+1} = F(x_t, H; H) \geq \overline{e}(x_t; H) \geq x_t$ where the second inequality follows the fact $e_{H,1} = 1$. Without loss of generality, assume $e_{H,0} < 1$ which implies $1 > x_{t+1} > x_t$ if $x_t < 1$. Furthermore, $\lim_{t\to\infty} x_t = 1$ which implies $\overline{x} = 1$. A contradiction.

Suppose $\underline{x} = \overline{x}$. This implies $\frac{\rho_{L,1}}{\rho_{L,0}+\rho_{L,1}} = \frac{\rho_{H,1}}{\rho_{H,0}+\rho_{H,1}}$ and equivalently $\frac{(1-p)e_0}{(1-q)(1-e_1)+(1-p)e_0} = \frac{pe_0}{q(1-e_1)+pe_0}$. The equality holds only if $e_0 = 0$ which implies $e_{L,0} = 0$ or $e_1 = 1$ which implies $e_{H,1} = 1$. A contradiction

(c) Consider an auxiliary optimization problem by imposing $e_{L,0} \leq e_{L,1}$ as follows

$$\max_{e_{L,0},e_{L,1},e_{H,0},e_{H,1}} \left\{ \frac{e_0}{1-(e_1-e_0)} \right\}$$

subject to

$$\frac{1-\beta}{1-\beta(e_1-e_0)}(e_{H,0}-e_{L,0}) \leq y \tag{53}$$

$$\frac{1-\beta}{1-\beta(e_1-e_0)}(e_{H,1}-e_{L,1}) \geq y \tag{54}$$

$$e_{L,0} \leq e_{L,1} \tag{55}$$

$$e_{L,0}, e_{L,1}, e_{H,0}, e_{H,1} \in [0,1]. \tag{56}$$

Step 1: $e_{H,1} = 1$.

I have $e_{H,1} > e_{L,1} \geq e_{L,0}$ where the first inequality follows (54) and the second follows (55). Furthermore, (53) and (54) imply $e_{H,1} - e_{L,1} \geq e_{H,0} - e_{L,0}$. So I conclude $e_{H,1} \geq e_{H,0}$. Hence, if $e_{H,1} < 1$, e' with $e'_\gamma = e_\gamma + (1 - e_{H,1})$ $\forall \gamma \in \Gamma$ satisfies (53)-(56) which is strictly more efficient.

Step 2: $e_{L,0} = e_{H,0} \leq e_{L,1}$ or $e_{L,0} = e_{L,1} \leq e_{H,0}$.

If $e_0 \leq e_{L,1}$, then $e' = (e_0 \ e_{L,1} \ e_0 \ e_{H,1})$ satisfies (53)-(56) and is as efficient as e. If $e_0 > e_{L,1}$, then $e' = (e_{L,1} \ e_{L,1} \ \frac{e_0-(1-q)e_{L,1}}{q} \ e_{H,1})$ satisfies (53)-(56) and is as efficient as e. Hence, without loss of generality, I focus on these two cases thereafter.

Step 3: Suppose $e_{L,0} = e_{H,0} \equiv e_0 \leq e_{L,1}$.

Notice (53) is satisfied automatically. Rewrite the optimization problem as follows,

$$\max_{e_0,e_{L,1}} \left\{ \frac{e_0}{1-[(1-p)e_{L,1}+p-e_0]} \right\}$$

subject to

$$\frac{1-\beta}{1-\beta[(1-p)e_{L,1}+p-e_0]}(1-e_{L,1}) \geq y \tag{57}$$

$$e_0 \leq e_{L,1} \tag{58}$$

$$e_0, e_{L,1} \in [0,1]. \tag{59}$$

Since the left hand side of (57) is decreasing in $e_{L,1}$, then (58) must bind. Hence, I have

$$e_0 = (1-p)e_{L,1} + p - \left[\frac{1}{\beta} - \frac{(1-\beta)(1-e_{L,1})}{\beta y}\right] \text{ with } \frac{de_0}{de_{L,1}} \leq 0.$$

Inserting it into the objective function gives

$$\frac{(1-p)e_{L,1}+p-\left[\frac{1}{\beta}-\frac{(1-\beta)(1-e_{L,1})}{\beta y}\right]}{1-\left[\frac{1}{\beta}-\frac{(1-\beta)(1-e_{L,1})}{\beta y}\right]}$$

which is decreasing in $e_{L,1}$. Hence, (57) must bind which degenerates into the case considered in Step 4.

Step 4: Suppose $e_{L,0}=e_{L,1}\equiv e_L\leq e_{H,0}$.

Rewrite the optimization problem as follows,

$$\max_{e_L,e_{H,0}}\left\{\frac{(1-q)e_L+qe_{H,0}}{1-[p-qe_{H,0}-(p-q)e_L]}\right\}$$

subject to

$$\frac{1-\beta}{1-\beta[p-qe_{H,0}-(p-q)e_L]}(e_{H,0}-e_L)\leq y \tag{60}$$

$$\frac{1-\beta}{1-\beta[p-qe_{H,0}-(p-q)e_L]}(1-e_L)\geq y \tag{61}$$

$$e_L\leq e_{H,0} \tag{62}$$

$$e_L,e_{H,0}\in[0,1]. \tag{63}$$

Notice (1) the left hand side of (60) is increasing in $e_{H,0}$ and decreasing in e_L; (2) the left hand side of (61) is decreasing in both $e_{H,0}$ and e_L. Furthermore, $e_L=\frac{1}{\beta}\frac{c}{(p-q)^2(\theta_H-\theta_L)}$ and $e_{H,0}=1$ make both (60) and (61) bind. So I conclude (61) binds, but (60) does not. Hence,

$$p-qe_{H,0}-(p-q)e_L=\frac{1}{\beta}-\frac{(1-\beta)(1-e_L)}{\beta y}\text{ with }\frac{de_L}{de_{H,0}}\leq 0.$$

Inserting it into the objective function gives

$$\frac{p+(1-p)e_L-\left[\frac{1}{\beta}-\frac{(1-\beta)(1-e_L)}{\beta y}\right]}{1-\left[\frac{1}{\beta}-\frac{(1-\beta)(1-e_L)}{\beta y}\right]}$$

which is decreasing in e_L. Then it is optimal to have $e_L=\frac{1}{\beta}\frac{c}{(p-q)^2(\theta_H-\theta_L)}$ and $e_{H,0}=1$.

So I conclude that there does not exist a decision function supporting truth-reporting which features $e_{L,0}\leq e_{L,1}$ and is strictly more efficient than e^*.

This completes the proof.

1.5.14 Proof of Proposition 1.9

By Proposition 1.8, I have

$$\mathbb{N}_{t+1} = \overline{\Pi}(\mathbb{N}_t) \equiv \left\{ x \in [0,1] \;\middle|\; \begin{array}{c} \exists \lambda \in \Lambda \text{ and } \theta \in \Theta \\ \text{such that} \\ x = F(x', \theta; \lambda) \text{ for some } x' \in \mathbb{N}_t \end{array} \right\} \; \forall t \geq 1.$$

Given $\mathbb{N}_0(L)$ and $\mathbb{N}_0(H)$ are singletons, $\mathbb{N}_1$ is finite because both Λ and Θ are finite. By induction, I conclude $\mathbb{N}_t$ is finite for $t \geq 1$. Let $\underline{x}_t \equiv \min \mathbb{N}_t$ and $\overline{x}_t \equiv \max \mathbb{N}_t$ for $t \geq 1$. I have

$$\underline{x}_{t+1} = \min F(x, \theta; \lambda) \text{ s.t. } x \in \{\underline{x}_t, \overline{x}_t\}, \theta \in \Theta \text{ and } \lambda \in \Lambda \tag{64}$$

$$\overline{x}_{t+1} = \max F(x, \theta; \lambda) \text{ s.t. } x \in \{\underline{x}_t, \overline{x}_t\}, \theta \in \Theta \text{ and } \lambda \in \Lambda \tag{65}$$

because $F(x, \theta; \lambda)$ is monotone in x. By definition, there exist $\lambda \in \Lambda$ and $\theta \in \Theta$ such that $\underline{x}_1 = F\left(\frac{\rho_{\lambda,1}}{\rho_{\lambda,0}+\rho_{\lambda,1}}, \theta; \lambda\right)$. Given $\underline{x}_1 \leq \frac{\rho_{\lambda,1}}{\rho_{\lambda,0}+\rho_{\lambda,1}} \leq \overline{x}_1$ as shown in Proposition 1.8, either $F(\underline{x}_1, \theta; \lambda)$ or $F(\overline{x}_1, \theta; \lambda)$ must be less than $\underline{x}_1$ because $F(x, \theta; \lambda)$ is monotone in x. Hence, $\underline{x}_2 \leq \underline{x}_1$. Similarly, there exist $\lambda \in \Lambda$ and $\theta \in \Theta$ such that $\overline{x}_1 = F\left(\frac{\rho_{\lambda,1}}{\rho_{\lambda,0}+\rho_{\lambda,1}}, \theta; \lambda\right)$. Given $\underline{x}_1 \leq \frac{\rho_{\lambda,1}}{\rho_{\lambda,0}+\rho_{\lambda,1}} \leq \overline{x}_1$ as shown in Proposition 1.8, either $F(\underline{x}_1, \theta; \lambda)$ or $F(\overline{x}_1, \theta; \lambda)$ must be greater than $\overline{x}_1$ because $F(x, \theta; \lambda)$ is monotone in x. Hence, $\overline{x}_2 \geq \overline{x}_1$. By induction, I conclude $\{\underline{x}_t\}_{t=1}^{\infty}$ is a decreasing sequence with $\lim_{t\to\infty} \underline{x}_t = \underline{x}$ and $\{\overline{x}_t\}_{t=1}^{\infty}$ is an increasing sequence with $\lim_{t\to\infty} \overline{x}_t = \overline{x}$.

By taking t to ∞ in (64) and (65), I have

$$\underline{x} = \min F(x, \theta; \lambda) \text{ s.t. } x \in \{\underline{x}, \overline{x}\}, \theta \in \Theta \text{ and } \lambda \in \Lambda$$

$$\overline{x} = \max F(x, \theta; \lambda) \text{ s.t. } x \in \{\underline{x}, \overline{x}\}, \theta \in \Theta \text{ and } \lambda \in \Lambda$$

because $F(x, \theta; \lambda)$ is continuous in x. Furthermore, given $F(x, L; \lambda) \leq F(x, H; \lambda)$, the results follow.

I have

$$\overline{e}(x; L) = e_{L,0} + (e_{L,1} - e_{L,0})x \leq e_{H,0} + (e_{H,1} - e_{H,0})x = \overline{e}(x; H) \; \forall x \in [0,1]$$

because $e_{L,0} \leq e_{H,0}$ and $e_{L,1} - e_{L,0} \leq e_{H,1} - e_{H,0}$ implied by $e_{H,0} - e_{L,0} \leq e_{H,1} - e_{L,1}$. Hence,

$$F(x, \theta; L) \leq F(x, \theta; H) \; \forall x \in [0,1] \text{ and } \theta \in \Theta.$$

which implies

$$\underline{x} = \min F(x, L; L) s.t. x \in \{\underline{x}, \overline{x}\}$$

$$\overline{x} = \max F(x, H; H) s.t. x \in \{\underline{x}, \overline{x}\}.$$

Furthermore, since $e_{L,0} \geq e_{L,1}$ and $\underline{x} \leq \overline{x}$, I have $\overline{e}(\underline{x}; L) \geq \overline{e}(\overline{x}; L)$ which implies $F(\underline{x}, L; L) \geq F(\overline{x}, L; L)$. So $\underline{x} = F(\overline{x}, L; L)$. (a) If $e_{H,0} \leq e_{H,1}$, then $\overline{e}(\underline{x}; H) \leq \overline{e}(\overline{x}; H)$ which implies $F(\underline{x}, H; H) \leq F(\overline{x}, H; H)$. So $\overline{x} = F(\overline{x}, H; H)$; (b) If $e_{H,0} \geq e_{H,1}$, then $\overline{e}(\underline{x}; H) \geq \overline{e}(\overline{x}; H)$ which implies $F(\underline{x}, H; H) \geq F(\overline{x}, H; H)$. So $\overline{x} = F(\underline{x}, H; H)$.

1.5.15 Proof of Theorem 1.4

I proceed by considering one case with $e^+_{H,0} \leq e^+_{H,1}$ and one case with $e^+_{H,0} \geq e^+_{H,1}$.

Case 1: Suppose $e^+_{H,0} \leq e^+_{H,1}$. Hence,

$$e^+ \in \underset{e_{L,0}, e_{L,1}, e_{H,0}, e_{H,1}}{\arg\max} \left\{ \frac{e_0}{1 - (e_1 - e_0)} \right\}$$

subject to

$$(1 - F(x, L; \lambda))(\mathbf{e}_{H,0} - \mathbf{e}_{L,0}) + F(x, L; \lambda)(\mathbf{e}_{H,1} - \mathbf{e}_{L,1}) \leq y \ \forall \lambda \in \Lambda \text{ and } x \in \{\underline{x}, \overline{x}\} \quad (66)$$

$$(1 - F(x, H; \lambda))(\mathbf{e}_{H,0} - \mathbf{e}_{L,0}) + F(x, H; \lambda)(\mathbf{e}_{H,1} - \mathbf{e}_{L,1}) \geq y \ \forall \lambda \in \Lambda \text{ and } x \in \{\underline{x}, \overline{x}\} \quad (67)$$

$$e_{L,0} \geq e_{L,1} \quad (68)$$

$$e_{L,0} \leq e_{H,0} \quad (69)$$

$$e_{H,0} \leq e_{H,1} \quad (70)$$

$$e_{H,0} - e_{L,0} \leq e_{H,1} - e_{L,1} \quad (71)$$

$$e_{L,0}, e_{L,1}, e_{H,0}, e_{H,1} \in [0, 1] \quad (72)$$

where $\{\underline{x}, \overline{x}\}$ solve

$$\underline{x} = F(\overline{x}, L; L) \quad (73)$$

$$\overline{x} = F(\overline{x}, H; H). \quad (74)$$

Notice (66) and (67) are no profitable one-shot deviations conditions as usual, (68) follows Theorem 1.3, (69) follows $e^+_{L,0} \leq e^+_{H,0}$, (70) is imposed by assumption and (71) follows Proposition 1.5. Furthermore, given (68)-(71), (73) and (74) follow (b) in Proposition 1.9.

Step 1: Since e^+ is strictly more efficient than the optimal belief-free decision function e^*, (71) does not bind which implies (66) and (67) can be rewritten as

$$F(x, L; \lambda) \leq \mathbf{F} \leq F(x, H; \lambda) \ \forall \lambda \in \Lambda \text{ and } x \in \{\underline{x}, \overline{x}\}. \quad (75)$$

Furthermore, I have

$$F(\overline{x}, L; H) \geq F(\underline{x}, L; H) \geq F(\underline{x}, L; L) \geq F(\overline{x}, L; L)$$

$$F(\overline{x}, H; L) \leq F(\underline{x}, H; L) \leq F(\underline{x}, H; H) \leq F(\overline{x}, H; H)$$

by (**??**)-(71) and $\underline{x} \leq \overline{x}$. So (75) can be further simplified as

$$F(\overline{x}, L; H) \leq \mathbf{F} \leq F(\overline{x}, H; L). \tag{76}$$

Notice Theorem 1.4 follows if both constraints in (76) bind.

Step 2: $F(\overline{x}^{+}, L; H) = \mathbf{F}^{+}$.

I show $\sum_{\gamma\in\Gamma} \frac{d\overline{e}(\overline{x};L)}{de_\gamma}\Big|_{e^+} > 0$ which implies $\sum_{\gamma\in\Gamma} \frac{dF(\overline{x},H;L)}{de_\gamma}\Big|_{e^+} > 0$ because $F(\overline{x}, H; L)$ is strictly increasing in $\overline{e}(\overline{x}; L)$.

Therefore, a contradiction can constructed if $F(\overline{x}^{+}, L; H) < \mathbf{F}^{+} \leq F(\overline{x}^{+}, H; L)$ as follows: there exists $\varepsilon > 0$ such that e' with $e'_\gamma = e^+_\gamma + \varepsilon\ \forall\gamma \in \Gamma$ satisfies (68)-(72) because $e^+_{L,1} \leq e^+_{L,0} \leq e^+_{H,0} \leq e^+_{H,1} < 1$ by (68)-(70). Additionally,

$$F(\overline{x}', L; H) < \mathbf{F}' = \mathbf{F}^{+} \leq F(\overline{x}^{+}, H; L) < F(\overline{x}', H; L)$$

which implies e' supports truth-reporting and is strictly more efficient than e^+.

Rewrite (73) and (74) as

$$(p-q)\overline{e}(\overline{x}; L)\overline{e}(\overline{x}; H) + q\overline{e}(\overline{x}; L) + (qe_{L,0} - pe_{L,1})\overline{e}(\overline{x}; H) - qe_{L,0} = 0 \tag{77}$$

$$(p-q)(\overline{e}(\overline{x}; H))^2 + [(qe_{H,0} - pe_{H,1}) + q]\overline{e}(\overline{x}; H) - qe_{H,0} = 0. \tag{78}$$

Totally differentiating (77) and (78) gives

$$A \begin{bmatrix} d\overline{e}(\overline{x}; L) \\ d\overline{e}(\overline{x}; H) \end{bmatrix} = \begin{bmatrix} q(1-\overline{e}(\overline{x}; H)) & p\overline{e}(\overline{x}; H) & 0 & 0 \\ 0 & 0 & q(1-\overline{e}(\overline{x}; H)) & p\overline{e}(\overline{x}; H) \end{bmatrix} \begin{bmatrix} de_{L,0} \\ de_{L,1} \\ de_{H,0} \\ de_{H,1} \end{bmatrix} \tag{79}$$

where

$$A = \begin{bmatrix} (p-q)\overline{e}(\overline{x}; H) + q & (p-q)\overline{e}(\overline{x}; L) + (qe_{L,0} - pe_{L,1}) \\ 0 & 2(p-q)\overline{e}(\overline{x}; H) + (qe_{H,0} - pe_{H,1}) + q \end{bmatrix}.$$

Let $de_{L,0} = de_{L,1} = de_{H,0} = de_{H,1} = \varepsilon > 0$. By (79), I have

$$A \begin{bmatrix} d\overline{e}(\overline{x};L) \\ d\overline{e}(\overline{x};H) \end{bmatrix} = \begin{bmatrix} q(1-\overline{e}(\overline{x};H)) + p\overline{e}(\overline{x};H) \\ q(1-\overline{e}(\overline{x};H)) + p\overline{e}(\overline{x};H) \end{bmatrix} \varepsilon \tag{80}$$

which implies

$$d\overline{e}(\overline{x};L)|_{e^+} = \frac{A_{22}-A_{12}}{A_{11}A_{22}}[q(1-\overline{e}(\overline{x};H)) + p\overline{e}(\overline{x};H)]\varepsilon\Big|_{e^+}.$$

Furthermore,

$$A_{11}|_{e^+} = (p-q)\overline{e}(\overline{x}^+;H) + q > 0$$

$$A_{22}|_{e^+} = 2(p-q)\overline{e}(\overline{x}^+;H) + (qe^+_{H,0} - pe^+_{H,1}) + q = (p-q)\overline{e}(\overline{x}^+;H) + \frac{qe^+_{H,0}}{\overline{e}(\overline{x}^+;H)} > 0$$

by (78) and

$$\begin{aligned} & A_{22} - A_{12}|_{e^+} \\ =& 2(p-q)\overline{e}(\overline{x}^+;H) + (qe^+_{H,0} - pe^+_{H,1}) + q - (p-q)\overline{e}(\overline{x}^+;L) - (qe^+_{L,0} - pe^+_{L,1}) \\ >& (p-q)\overline{e}(\overline{x}^+;H) + (qe^+_{H,0} - pe^+_{H,1}) + q - (qe^+_{L,0} - pe^+_{L,1}) \\ =& \frac{qe^+_{H,0}}{\overline{e}(\overline{x}^+;H)} - (qe^+_{L,0} - pe^+_{L,1}) \\ \geq& q(e^+_{H,0} - e^+_{L,0}) + pe^+_{L,1} \geq 0 \end{aligned}$$

where the first inequality follows

$$\overline{e}(\overline{x}^+;L) = e^+_{L,0} + (e^+_{L,1} - e^+_{L,0})\overline{x}^+ < e^+_{H,0} + (e^+_{H,1} - e^+_{H,0})\overline{x}^+ = \overline{e}(\overline{x}^+;H)$$

because $e^+_{L,0} \leq e^+_{H,0}$ by (69) and $e^+_{L,1} - e^+_{L,0} < e^+_{H,1} - e^+_{H,0}$ by the fact e^+ is belief-based and $\overline{x}^+ \in (0,1)$ by Proposition 11, the second equality follows (78), the second inequality follows $\overline{e}(\overline{x}^+;H) \leq 1$, the last inequality follows (69).

So $d\overline{e}(\overline{x};L)|_{e^+} > 0$ for $de_{L,0} = de_{L,1} = de_{H,0} = de_{H,1} = \varepsilon > 0$ which implies $\sum_{\gamma\in\Gamma} \frac{d\overline{e}(\overline{x};L)}{de_\gamma}\Big|_{e^+} > 0$.

Step 3: If $\mathbf{F}^+ < F(\overline{x}^+, H; L)$, then $e^+_{H,0} = e^+_{H,1}$ which will be considered in Case 2 below.

I show $\frac{d\mathbf{F}}{de_{L,1}} - \frac{1-p}{p}\frac{d\mathbf{F}}{de_{H,1}}\Big|_{e^+} > 0$ (by algebra) and $\frac{de(\overline{x};H)}{de_{L,1}} - \frac{1-p}{p}\frac{de(\overline{x};H)}{de_{H,1}}\Big|_{e^+} < 0$. Therefore, a contradiction can be constructed if $\mathbf{F}^+ < F(\overline{x}^+, H; L)$ and $e^+_{H,0} < e^+_{H,1}$ as follows: there exists $\varepsilon > 0$ such that $e' = \left(e^+_{L,0}\ e^+_{L,1} + \varepsilon\ e^+_{H,0}\ e^+_{H,1} - \frac{1-p}{p}\varepsilon\right)$ satisfies (68)-(72) because (68) does not bind at e^+ by Proposition 11 and (71) does not bind at e^+ by assumption. Additionally,

$$F(\overline{x}', L; H) < F(\overline{x}^+, L; H) = \mathbf{F}^+ < \mathbf{F}' < F(\overline{x}', H; L)$$

which implies e' supports truth-reporting and is as efficient as e^+ because $e'_0 = e^+_0$ and $e'_1 = e^+_1$ by construction. But since both constraints have been relaxed, a strictly more efficient decision function supporting truth-reporting can be constructed accordingly.

Let $de_{L,0} = de_{H,0} = 0$, $de_{L,1} = \varepsilon$ and $de_{H,1} = -\frac{1-p}{p}\varepsilon$ for $\varepsilon > 0$. By (79), I have

$$A \begin{bmatrix} d\overline{e}(\overline{x}; L) \\ d\overline{e}(\overline{x}; H) \end{bmatrix} = \begin{bmatrix} p\overline{e}(\overline{x}; H) \\ -(1-p)\overline{e}(\overline{x}; H) \end{bmatrix} \varepsilon \tag{81}$$

which implies

$$d\overline{e}(\overline{x}; H)|_{e^+} = -\frac{(1-p)\overline{e}(\overline{x}; H)}{A_{22}}\varepsilon\bigg|_{e^+} < 0$$

because $A_{22}|_{e^+} > 0$ as proven in Step 2.

Case 2: Suppose $e^+_{H,0} \geq e^+_{H,1}$. Hence,

$$e^+ \in \underset{e_{L,0}, e_{L,1}, e_{H,0}, e_{H,1}}{\arg\max} \left\{ \frac{e_0}{1 - (e_1 - e_0)} \right\}$$

subject to

$$(1 - F(x, L; \lambda))(\mathbf{e}_{H,0} - \mathbf{e}_{L,0}) + F(x, L; \lambda)(\mathbf{e}_{H,1} - \mathbf{e}_{L,1}) \leq y \ \forall \lambda \in \Lambda \text{ and } x \in \{\underline{x}, \overline{x}\} \tag{82}$$

$$(1 - F(x, H; \lambda))(\mathbf{e}_{H,0} - \mathbf{e}_{L,0}) + F(x, H; \lambda)(\mathbf{e}_{H,1} - \mathbf{e}_{L,1}) \geq y \ \forall \lambda \in \Lambda \text{ and } x \in \{\underline{x}, \overline{x}\} \tag{83}$$

$$e_{L,0} \geq e_{L,1} \tag{84}$$

$$e_{L,0} \leq e_{H,0} \tag{85}$$

$$e_{H,0} \geq e_{H,1} \tag{86}$$

$$e_{H,0} - e_{L,0} \leq e_{H,1} - e_{L,1} \tag{87}$$

$$e_{L,0}, e_{L,1}, e_{H,0}, e_{H,1} \in [0, 1] \tag{88}$$

where $\{\underline{x}, \overline{x}\}$ solve

$$\underline{x} = F(\overline{x}, L; L) \tag{89}$$

$$\overline{x} = F(\underline{x}, H; H). \tag{90}$$

Notice (82) and (83) are no profitable one-shot deviations conditions as usual, (84) follows Theorem 1.3, (85) follows $e^+_{L,0} \leq e^+_{H,0}$, (86) is imposed by assumption and

(87) follows Proposition 1.5. Furthermore, given (84)-(87), (89) and (90) follow (a) in Proposition 1.9.

Step 1: Since e^+ is strictly more efficient than the optimal belief-free decision function e^*, (87) does not bind which implies (82) and (83) can be rewritten as

$$F(x, L; \lambda) \leq \mathbf{F} \leq F(x, H; \lambda) \ \forall \lambda \in \Lambda \text{ and } x \in \{\underline{x}, \overline{x}\}. \tag{91}$$

Furthermore, I have

$$F(\underline{x}, L; H) \geq F(\underline{x}, L; L) \text{ and } F(\underline{x}, L; H) \geq F(\overline{x}, L; H) \geq F(\overline{x}, L; L)$$

$$F(\overline{x}, H; L) \leq F(\overline{x}, H; H) \text{ and } F(\overline{x}, H; L) \leq F(\underline{x}, H; L) \leq F(\underline{x}, H; H)$$

by (84)-(87) and $\underline{x} \leq \overline{x}$. So (91) can be further simplified as

$$F(\underline{x}, L; H) \leq \mathbf{F} \leq F(\overline{x}, H; L). \tag{92}$$

Notice Proposition 13 follows if both constraints in (92) bind.

Step 2: $F(\underline{x}^+, L; H) = \mathbf{F}^+$.

I show $\sum_{\gamma \in \Gamma} \frac{d\overline{e}(\overline{x};L)}{de_\gamma}\Big|_{e^+} > 0$ which implies $\sum_{\gamma \in \Gamma} \frac{dF(\overline{x},H;L)}{de_\gamma}\Big|_{e^+} > 0$ because $F(\overline{x}, H; L)$ is strictly increasing in $\overline{e}(\overline{x}; L)$. Therefore, a contradiction can be constructed if $F(\underline{x}^+, L; H) < \mathbf{F}^+$ as follows: there exists $\varepsilon > 0$ such that e' with $e'_\gamma = e^+_\gamma + \varepsilon$ $\forall \gamma \in \Gamma$ satisfies (84)-(88). (Here, I implicitly assume $e^+_{H,0} < 1$. Otherwise, I have $e^+_{H,0} = 1 > e^+_{H,1}$ so that $e' = (e^+_{L,0} \ e^+_{L,1} + \varepsilon \ e^+_{H,0} \ e^+_{H,1} + \varepsilon)$ for some $\varepsilon > 0$ works similarly because I can show $\frac{d\mathbf{F}}{de_{L,1}} + \frac{d\mathbf{F}}{de_{H,1}}\Big|_{e^+} < 0$ and $\frac{d\overline{e}(\overline{x};L)}{de_{L,1}} + \frac{d\overline{e}(\overline{x};L)}{de_{H,1}}\Big|_{e^+} > 0$ which are left for the readers.) Additionally,

$$F(\underline{x}', L; H) < \mathbf{F}' = \mathbf{F}^+ \leq F(\overline{x}^+, H; L) < F(\overline{x}', H; L)$$

which implies e' supports truth-reporting and is strictly more efficient than e^+.

Rewrite (89) and (90) as

$$(p - q)\overline{e}(\overline{x}; L)\overline{e}(\underline{x}; H) + q\overline{e}(\overline{x}; L) + (qe_{L,0} - pe_{L,1})\overline{e}(\underline{x}; H) - qe_{L,0} = 0 \tag{93}$$

$$-(p-q)\overline{e}(\overline{x}; L)\overline{e}(\underline{x}; H) + [(1-q)e_{H,0} - (1-p)e_{H,1}]\overline{e}(\overline{x}; L) + (1-q)\overline{e}(\underline{x}; H) - (1-q)e_{H,0} = 0. \tag{94}$$

Totally differentiating (93) and (94) gives

$$A\begin{bmatrix} d\overline{e}(\overline{x};L) \\ d\overline{e}(\underline{x};H)\end{bmatrix} = \begin{bmatrix} q(1-\overline{e}(\underline{x};H)) & p\overline{e}(\underline{x};H) & 0 & 0 \\ 0 & 0 & (1-q)(1-\overline{e}(\overline{x};L)) & (1-p)\overline{e}(\overline{x};L)\end{bmatrix}\begin{bmatrix} de_{L,0} \\ de_{L,1} \\ de_{H,0} \\ de_{H,1}\end{bmatrix} \tag{95}$$

where

$$A = \begin{bmatrix} (p-q)\overline{e}(\underline{x};H)+q & (p-q)\overline{e}(\overline{x};L)+(qe_{L,0}-pe_{L,1}) \\ -(p-q)\overline{e}(\underline{x};H)+[(1-q)e_{H,0}-(1-p)e_{H,1}] & -(p-q)\overline{e}(\overline{x};L)+(1-q)\end{bmatrix}.$$

Let $de_{L,0} = de_{L,1} = de_{H,0} = de_{H,1} = \varepsilon > 0$. By (95), I have

$$A\begin{bmatrix} d\overline{e}(\overline{x};L) \\ d\overline{e}(\underline{x};H)\end{bmatrix} = \begin{bmatrix} (p-q)\overline{e}(\underline{x};H)+q \\ -(p-q)\overline{e}(\overline{x};L)+(1-q)\end{bmatrix}\varepsilon \tag{96}$$

which implies

$$d\overline{e}(\overline{x};L)|_{e^+} = \left.\frac{[-(p-q)\overline{e}(\overline{x};L)+(1-q)][(p-q)\overline{e}(\underline{x};H)+q-(p-q)\overline{e}(\overline{x};L)-(qe_{L,0}-pe_{L,1})]}{\|A\|}\varepsilon\right|_{e^+}$$

where the numerator evaluated at e^+ is strictly positive given $\overline{e}(\overline{x}^+;L) < \overline{e}(\underline{x}^+;H)$ so that it suffices to show $\|A\||_{e^+} > 0$. Furthermore, I have

$$A_{11}|_{e^+}, A_{12}|_{e^+}, A_{22}|_{e^+} > 0, \;\; A_{21}|_{e^+} \geq 0, \;\; A_{11}|_{e^+} > A_{12}|_{e^+} \text{ and } A_{22}|_{e^+} > A_{21}|_{e^+}$$

so that $\|A\||_{e^+} = A_{11}|_{e^+} A_{22}|_{e^+} - A_{12}|_{e^+} A_{21}|_{e^+} > 0$.

Step 3: $\mathbf{F}^+ = F(\overline{x}^+, H; L)$.

I show $\left.\frac{d\mathbf{F}}{de_{L,1}} - \frac{1-p}{p}\frac{d\mathbf{F}}{de_{H,1}}\right|_{e^+} > 0$ (by algebra) and $\left.\frac{de(\underline{x};H)}{de_{L,1}} - \frac{1-p}{p}\frac{de(\underline{x};H)}{de_{H,1}}\right|_{e^+} < 0$. Therefore, a contradiction can be constructed if $\mathbf{F}^+ < F(\overline{x}^+, H; L)$ as follows: there exists $\varepsilon > 0$ such that $e' = \left(e^+_{L,0}\; e^+_{L,1} + \varepsilon\; e^+_{H,0}\; e^+_{H,1} - \frac{1-p}{p}\varepsilon\right)$ satisfies (68)-(72) because (84) and (87) do not bind at e^+. Additionally,

$$F(\underline{x}', L; H) < F(\underline{x}^+, L; H) = \mathbf{F}^+ < \mathbf{F}' < F(\overline{x}', H; L)$$

which implies e' supports truth-reporting and is as efficient as e^+ because $e'_0 = e^+_0$ and $e'_1 = e^+_1$ by construction. But since both constraints have been relaxed, a strictly more efficient decision function supporting truth-reporting can be constructed accordingly.

Let $de_{L,0} = de_{H,0} = 0$, $de_{L,1} = \varepsilon$ and $de_{H,1} = -\frac{1-p}{p}\varepsilon$ for $\varepsilon > 0$. By (95), I have

$$A \begin{bmatrix} d\overline{e}(\overline{x}; L) \\ d\overline{e}(\underline{x}; H) \end{bmatrix} = \begin{bmatrix} p\overline{e}(\underline{x}; H) \\ -\frac{(1-p)^2}{p}\overline{e}(\overline{x}; L) \end{bmatrix} \varepsilon \tag{97}$$

which implies

$$d\overline{e}(\underline{x}; H)|_{e^+} = -\frac{p\overline{e}(\underline{x}; H)A_{21} + \frac{(1-p)^2}{p}\overline{e}(\overline{x}; L)A_{11}}{\|A\|}\varepsilon\Bigg|_{e^+}.$$

Since $A_{11}|_{e^+} > 0$,

$$\begin{aligned} & A_{21}|_{e^+} \\ =& -(p-q)\overline{e}(\underline{x}^+; H) + (1-q)e^+_{H,0} - (1-p)e^+_{H,1} \geq -(p-q)\overline{e}(\underline{x}^+; H) + (p-q)e^+_{H,0} > 0 \end{aligned}$$

by $e^+_{H,0} \geq e^+_{H,1}$ and $\|A\||_{e^+} > 0$ as proven in Step 2, $d\overline{e}(\underline{x}; H)|_{e^+} < 0$.

This completes the proof.

1.5.16 Proof of Lemma 1.3

Given $\mathbf{F}^+ \in (0,1)$, I can solve for $\overline{e}(\overline{x}^+; L)$ and $\overline{e}(\underline{x}^+; L)$ as

$$\overline{e}(\overline{x}^+; L) = \frac{q\mathbf{F}^+}{-(p-q)\mathbf{F}^+ + p} \text{ and } \overline{e}(\underline{x}^+; H) = \frac{(1-q)\mathbf{F}^+}{(p-q)\mathbf{F}^+ + (1-p)}$$

by $F(\underline{x}^+, L; H) = \mathbf{F}^+ = F(\overline{x}^+, H; L)$ as proved in Theorem 1.4. Taking $\mathbf{F}^+$, $\overline{e}(\overline{x}^+; L)$ and $\overline{e}(\underline{x}^+; H)$ as given, let's consider the set of the decision functions satisfying (93), (94) and

$$\frac{y - (\mathbf{e}_{H,0} - \mathbf{e}_{L,0})}{(\mathbf{e}_{H,1} - \mathbf{e}_{L,1}) - (\mathbf{e}_{H,0} - \mathbf{e}_{L,0})} = \mathbf{F}^+.$$

Notice each decision function in this set supports truth-reporting if it satisfies (84)-(88) as well. Rewriting these three equations gives

$$A \begin{bmatrix} e_{L,1} \\ e_{H,0} \\ e_{H,1} \end{bmatrix} = Be_{L,0} + C$$

where

$$A = \begin{bmatrix} -p\overline{e}(\underline{x}^+; H) & 0 & 0 \\ 0 & -(1-q)(1-\overline{e}(\overline{x}^+; L)) & -(1-p)\overline{e}(\overline{x}^+; L) \\ -\beta(1-p)y + (1-\beta)\mathbf{F}^+ & \beta qy - (1-\beta)(1-\mathbf{F}^+) & -\beta py - (1-\beta)\mathbf{F}^+ \end{bmatrix}$$

$$B = \begin{bmatrix} q(1-\overline{e}(\underline{x}^+; H)) \\ 0 \\ -[\beta(1-q)y + (1-\beta)(1-\mathbf{F}^+)] \end{bmatrix}$$

$$C = \begin{bmatrix} -[q + p(1-q)]\overline{e}(\overline{x}^+; L) + q(1-p)\overline{e}(\underline{x}^+; H) \\ p(1-q)\overline{e}(\overline{x}^+; L) - [(1-q) + q(1-p)]\overline{e}(\underline{x}^+; H) \\ -y \end{bmatrix}.$$

By algebra, I have

$$\begin{bmatrix} \frac{de_{L,1}}{de_{L,0}} \\ \frac{de_{H,0}}{de_{L,0}} \\ \frac{de_{H,1}}{de_{L,0}} \end{bmatrix} = A^{-1}B = \begin{bmatrix} -\frac{q(1-p)(1-\mathbf{F}^+)}{p(1-q)\mathbf{F}^+} \\ -\frac{q(1-p)}{p(1-q)} \frac{\beta q(1-p)^2 y(1-\mathbf{F}^+) + \beta p(1-q)^2 y\mathbf{F}^+ + (1-\beta)(p-q)(1-\mathbf{F}^+)\mathbf{F}^+}{\beta p^2(1-q)y(1-\mathbf{F}^+) + \beta q^2(1-p)y\mathbf{F}^+ + (1-\beta)(p-q)(1-\mathbf{F}^+)\mathbf{F}^+} \\ \frac{1-\mathbf{F}^+}{\mathbf{F}^+} \frac{\beta q(1-p)^2 y(1-\mathbf{F}^+) + \beta p(1-q)^2 y\mathbf{F}^+ + (1-\beta)(p-q)(1-\mathbf{F}^+)\mathbf{F}^+}{\beta p^2(1-q)y(1-\mathbf{F}^+) + \beta q^2(1-p)y\mathbf{F}^+ + (1-\beta)(p-q)(1-\mathbf{F}^+)\mathbf{F}^+} \end{bmatrix}$$

which implies

$$\frac{de_{L,1}}{de_{L,0}} < 0, \; \frac{de_{H,0}}{de_{L,0}} < 0 \text{ and } \frac{de_{H,1}}{de_{L,0}} > 0.$$

Furthermore, let $de_{L,0} = \varepsilon > 0$, I have

$$de_0 = \frac{1-\mathbf{F}^+}{p(1-q)} \frac{\beta[p^3(1-q)^3 - q^3(1-p)^3]y + (1-\beta)(p-q)[p(1-q)^2 - q^2(1-p)]\mathbf{F}^+}{\beta p^2(1-q)y(1-\mathbf{F}^+) + \beta q^2(1-p)y\mathbf{F}^+ + (1-\beta)(p-q)(1-\mathbf{F}^+)\mathbf{F}^+}\varepsilon$$

$$de_1 = \frac{1-\mathbf{F}^+}{p(1-q)} \frac{\beta[p^3(1-q)^3 - q^3(1-p)^3]y + (1-\beta)(p-q)[p^2(1-q) - q(1-p)^2](1-\mathbf{F}^+)}{\beta p^2(1-q)y(1-\mathbf{F}^+) + \beta q^2(1-p)y\mathbf{F}^+ + (1-\beta)(p-q)(1-\mathbf{F}^+)\mathbf{F}^+}\varepsilon$$

where $de_0 = (1-q)de_{L,0} + qde_{H,0}$ and $de_1 = (1-p)de_{L,1} + pde_{H,1}$. Hence,

$$\frac{de_0}{de_{L,0}} > 0 \text{ and } \frac{de_1}{de_{L,0}} > 0.$$

Therefore, a contradiction can be constructed if none of these three statements is true as follows: there exists $\varepsilon > 0$ such that e' with $e'_\gamma = e^+_\gamma + \frac{de_\gamma}{de_{L,0}}\varepsilon \; \forall \gamma \in \Gamma$ supports truth-reporting and is strictly more efficient than e^+ because $e'_0 > e^+_0$ and $e'_1 > e^+_1$.

This completes the proof.

1.5.17 Proof of Proposition 1.10

It suffices to prove that given any SE with respect to w^{**} in which the principal's strategy features truth-reporting, there exist $e, \mathbf{e}_{L,0}, \mathbf{e}_{L,1}, \mathbf{e}_{H,0}, \mathbf{e}_{H,1} \in [0,1]$ such that

(32)-(34) and

$$\frac{(1-e)(1-q)}{(1-e)(1-q)+e(1-p)}\mathbf{e}_{L,0}+\frac{e(1-p)}{(1-e)(1-q)+e(1-p)}\mathbf{e}_{L,1}$$

$$\frac{(1-e)q}{(1-e)q+ep}\mathbf{e}_{H,0}+\frac{ep}{(1-e)q+ep}\mathbf{e}_{H,1}$$

are the NED probabilities of the agent exerting effort in some SE with respect to w^{**} in which the principal's strategy features truth-reporting.

Suppose (e, r) is a SE with respect to w^{**} with the principal's truth-reporting. Define

$$e = e(\emptyset)$$

$$\mathbf{e}_\gamma = (1-\beta)\mathbb{E}_1\left[\sum_{t=1}^{\infty}\beta^{t-1}e_t\middle| s,\gamma\right] \ \forall\gamma$$

where $\mathbf{e}_\gamma$ is the NED probability of the agent at state γ at the beginning of period 1 exerting effort so that (32) is satisfied trivially and (33)-(34) are satisfied as the no profitable one-shot deviations conditions in period 0. Further define the agent's effort strategy e' as a mixed strategy with $e \mid (L,0)$ for the probability $\frac{(1-e_0)(1-q)}{(1-e_0)(1-q)+e_0(1-p)}$ and $e \mid (L,1)$ for the probability $\frac{e_0(1-p)}{(1-e_0)(1-q)+e_0(1-p)}$. I show $(e', r \mid (L,L))$ is a SE with respect to w^{**} in which $r \mid (L,L)$ features truth-reporting. $r \mid (L,L)$ features truth-reporting because (L,L) is still on the equilibrium path. No matter what the agent's history is, he assigns the probability one to the principal being at state (L,L) therefore having the continuation strategy $r \mid (L,L)$. That implies given $r \mid (L,L)$, both $e \mid (L,0)$ and $e \mid (L,1)$ show sequential rationality by Definition 1. So does e' as the mixed strategy. And given the principal's history (L,L), she assigns the probability $\frac{(1-e_0)(1-q)}{(1-e_0)(1-q)+e_0(1-p)}$ to the agent being at state $(L,0)$ therefore having the continuation strategy $e \mid (L,0)$. That implies given e', $r \mid (L,L)$ shows sequential rationality too by Definition 1. Therefore, the result follows.

A similar argument shows $(e'', r \mid (H,H))$ is a SE with respect to w^{**} in which e'' is a mixed strategy with $e \mid (H,0)$ for the probability $\frac{(1-e_0)q}{(1-e_0)q+e_0p}$ and $e \mid (H,1)$ for the probability $\frac{e_0q}{(1-e_0)q+e_0p}$ and $r \mid (H,H)$ features truth-reporting.

So, I have

$$(1-\beta)\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t e_t\middle| s\right] \in \Psi[0,1] \subseteq [0,1]$$

for any SE (e,r) with respect to w^{**} with the principal's truth-reporting. Proposition 10 follows by applying the logic above recursively. Notice the mapping Ψ is monotonic so that $\Psi^\infty([0,1])$ is well-defined.

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2 Equilibrium Matching and Termination

by Cheng Wang and Youzhi Yang

2.1 Introduction

In an equilibrium model of the labor market with moral hazard, we combine the theory of search and matching and the theory of dynamic contracting. Jobs are dynamic contracts. Equilibrium job separations are terminations of optimal dynamic contracts. Transitions from unemployment to new jobs are modelled as a process of matching and bargaining, as in Mortensen and Pissarides (1994). Matched workers and firms bargain over the values of the optimal contract to each party, and then the dynamics of the optimal contract will take them to a state of termination. Non-employed workers make consumption and saving decisions as in a typical growth model, but they must also decide whether or not to participate in the labor market. Firms enter freely into the market to endogenously determine the number of jobs in the economy.

The standard Mortensen-Pissarides equilibrium matching model of the labor market is built around two key assumptions: a matching and bargaining process sets the worker and firm pair up for an employment relationship, and a dynamic but exogenous process of match productivity then provides an engine for job separation. An important extension of the standard equilibrium matching model is Moscarini (2005), who puts the model of Jovanovic (1979) into the Mortensen-Pissarides framework to model separation as a process of learning about the productivity of the match, and allows the match to be dissolved once the perceived match productivity is sufficiently low.

We take a dynamic contract point of view to modelling equilibrium job separation in the Mortensen-Pissarides model. Workers and firms enter an optimal dynamic contract upon a match, and job separation is then modelled as the termination of the dynamic contract. In our environment of moral hazard, termination is used as an incentive device to induce worker efforts, and as a way of minimizing the cost of worker compensation. Workers that produce a sequence of bad outputs become too poor to motivate, and workers who produce a sequence of good outputs become too expensive to compensate and motivate, as in Spear and Wang (2005) and Sannikov (2008)[42]. Following a termination, workers are free to go back to the labor market to seek new matches, or choose to stay temporarily or permanently out of the labor

[42]Sannikov (2008) studies a continuous-time version of the dynamic principal-agent problem with optimal termination. He also establishes the result that optimal replacement occurs when the agent's continuation value is either too low or too high. He then analyzes how optimal termination depends on the parameters of the contracting environment, including for example the relative time preferences of the principal and the agent.

market. This generates equilibrium flows between employment and unemployment, flows from employment to not-in-the-labor-force, and flows from not-in-the-labor-force to unemployment.

Thus job separation is a purely endogenous process in our model, motivated by the dynamic provision of incentives and risk sharing. Workers and firms are homogeneous, and matches are identical: they operate the same production function in all periods. Termination occurs not because the technology of the match has evolved to be sufficiently poor as in Mortensen and Pissarides (1994), or it is found out to be sufficiently bad as in Moscarini (2005). Termination occurs because the economic relationship that evolved endogenously around the fixed match technology has become too costly for the parties to maintain.

This strategy we take in modelling the dynamics in the labor market allows us to determine endogenously and simultaneously the size and composition of all three states of the labor market: employment, unemployment, and not-in-the-labor-force, as well as the flows into and out of not-in-the-labor force. This is important, not only for the explanation of the economy's aggregate labor supply, but also necessary for providing a more coherent and complete view of the stocks and flows of the labor market. Labor market data, especially that in the recently available Current Population Survey, show significant flows of workers among all three states of the labor market, as documented in several recent researches including Fallick and Fleischman (2004), Nagypal (2005), Shimer (2005b). Yet most existing models in the search-matching literature of the labor market focus on the interaction between employment and unemployment (e.g., Mortensen and Pissarides (1997), Shimer (2005), Moscarini (2005), Nagypal (2005)), without modelling explicitly the state of not-in-the-labor-force and hence the size of the labor market. Sun-Bin Kim (2001) and Moscarini (2003) are exceptions. In both papers though, an additional source of worker heterogeneity is introduced into the Mortensen-Pissarises framework in order to generate flows into retirement. In Moscarini (2003) for example, the productivity of a match depends on a match specific variable, as well as a non-match-specific variable that captures the ability of the worker. The values of both variables are learned during a match, workers whose non-match-specific variable is learned to be sufficiently low choose to withdraw from the labor market.

Wang (2005) also models endogenous labor force participation, but without adding a second source of information friction or worker heterogeneity in addition to the one that motivates layoffs and unemployment. He considers a labor market with moral hazard and shows that, if one allows dynamic contracts to be optimally terminated in that environment, then terminations are of two types: involuntary layoffs and permanent retirements. In that paper, unemployment and non-participation are motivated by the same information friction, moral hazard, in a clean model environment with homogeneous workers and matches. Workers who have a sequence of low outputs become too costly to motivate and are laid off; workers who have performed well and are promised a sufficiently high expected utility become too expensive to motivate and compensate, they then retire permanently from the labor force.

Our approach to modelling the state of non-participation builds on Wang (2005) but adds an important new dimension: non-employed workers are allowed to make optimal consumption-saving decisions, and their decision on labor force participation is based on the amount of assets they hold. We show that a non-employed worker who holds a sufficiently small amount of assets will stay in the labor force after termination: their value outside the labor market is too low. We also show that a non-employed worker with a sufficiently large amount of saving will choose to quit the labor force permanently: their opportunity cost of re-entering the labor market is too high. In the quantitative version of our model that is calibrated to the U.S. data, there are non-employed workers whose level of wealth is neither sufficiently low to justify immediate returning back to the labor market, nor sufficiently high to justify staying permanently out of the labor market. These workers choose to quit the labor force temporarily, dissave, and eventually go back to the labor market once their assets are reduced to a sufficiently low level to make them profitable for the firms to employ. Thus our model generates not only flows into not-in-the-labor-force from employment, but also flows out of not-in-the-labor-force. Wang (2005) does not have equilibrium flows from non-participation to the labor force, neither does it explicitly model matching and bargaining in the labor market as we do in this paper.

That non-employed workers are allowed to save also distinguishes our model from the existing search-matching models. Most existing search-matching models of the labor market, including Mortensen and Pissarides (1994), do not allow workers to save. In those models, although employed workers are heterogeneous in the wages they earn, unemployed workers are homogeneous since they all hold zero amounts of assets. In our model, there is an equilibrium ergodic distribution of non-employed workers who differ in the amount of assets they hold. These workers make different consumption and saving decisions, and different decisions on labor force participation. While this endogenous wealth heterogeneity among non-employed workers gives rise to considerable technical complexities, it is essential to our model's being able to generate dynamics from not-in-the-labor-force to unemployment.

An important assumption in the Mortensen-Passarides model is that workers and firms cannot commit to long-term relationships, and wages are bargained sequentially upon realization of the current match productivity. In our model, firms can fully commit to any long-term contract, although workers are allowed to quit an ongoing employment relationship if it offers a value lower than the workers' outside alternative. Bargaining occurs only once, before the employment relationship begins, and is over the total values of the optimal contract to the parties. Wages are state contingent compensation payments to the worker that are dictated by the optimal structure of the optimal contract, not bargained repeatedly each period.

One could allow firms and workers to enter dynamic contracts and let bargaining take place only once also in Mortensen and Pissarides (1994). Without private information or other types of frictions though, perfect consumption smoothing would imply a constant wage over the worker's tenure at the firm. This in turn would imply a wage distribution that is essentially identical to the exogenous distribution of the random match productivity. The assumption of repeated wage bargaining, although helpful for generating an ergodic wage distribution in Mortensen and Pissarides (1994), implies that wages are not sufficiently rigid for the model to match business cycle movements in the data (Hall (2005), Shimer (2005)). Rudanko (2007) models long-term wage contracts with limited commitment in a search-matching model of the labor market to produce the observed wage rigidity/volatility. Obviously, our model offers a potential alternative for accounting for the observed wage rigidity/volatility that is based on an optimal trade off between consumption smoothing and incentives.

In an important recent study, Hornstein, Krusell and Violante (2006) argue that standard equilibrium search-matching models can generate only a very small, 3.6%, differential between the average and the lowest wages paid in the U.S. labor market, and the observed Mm ratio–the ratio between the average wage and lowest wage paid–is at least twenty times larger than what the model observes. As the paper explains, "The short unemployment durations, as in the U.S. data, reveal that agents in the model do not find it worthwhile to wait because frictional wage inequality is tiny. The message of search theory is that 'good things come to those who wait", so if the wait is short, it must be that good things are not likely to happen.' (page 9.) The paper further shows that the extensions of the standard search and matching models can only modestly improve their performance on accounting for the observed Mm ratio.

Our model is capable of generating much larger wage dispersions than the existing equilibrium search and matching models. In a quantitative version of our model that is calibrated to the U.S. data, the computed Mm ratio is 24.5, similar to what Hornstein, Krusell and Violante observe in the U.S. data. The reason our model is better in accounting for the observed wage dispersion is clear. In our model, wage dispersion is driven by the provision of intertemporal incentives and intertemporal risk sharing. Wages of homogenous workers who start with the same initial expected utility fan out over time as their outputs follow a stochastic process. In our model, workers who produce a sequence of high outputs will see their wages increase over time, and workers that produce a sequence of low outputs will see their wages decrease over time. This effect of the dynamic contracting on distribution was first discussed in Green (1987) and Atkeson and Lucas (1991). In this paper, the same mechanism is put to work in an equilibrium search/matching framework.

The model is presented in the next section. In Sections 2.3 and 2.4 we formulate and analyze a stationary equilibrium of the model. We then calibrate the model to the U.S. economy in Section 2.5 to study the structure of the equilibrium dynamic contract, the stocks and flows of the labor market, and worker compensation dynamics. Section 2.6 concludes the paper.

2.2 Setup

Let time be denoted $t = 1, 2, \cdots$ The model economy has one consumption good, and is populated by one unit of homogeneous workers. Workers survive into the next period with a constant probability $\Delta \in (0, 1)$. At the beginning of each period, $1 - \Delta$ units of workers are born so the measure of workers in each period is constant at one. Workers that are born in period $\tau(\geq 1)$ have the following preferences:

$$E_\tau \sum_{t=\tau}^{\infty} (\beta\Delta)^{t-\tau}[u(c_t) - \phi(a_t)],$$

where $\beta \in (0, 1)$ is the worker's discount factor, $u : \mathbb{R}_+ \to \mathbb{R}$ denotes the worker's utility function, c_t his consumption; $\phi : \{0\} \bigcup [\underline{a}, \overline{a}] \to \mathbb{R}$, where $\overline{a} > \underline{a} \geq 0$, denotes the worker's disutility function, a_t his effort. We make the following assumptions on u and ϕ: $u(\cdot)$ is bounded, continuous, differentiable, and strictly concave; $\phi(\cdot)$ is continuously differentiable, and strictly convex on $[\underline{a}, \overline{a}]$.

The model economy is also populated with a large measure of identical firms. Firms maximize expected discounted profits, and they discount future profits using a constant discount factor $1/(1 + r)$, where $r > 0$ denotes the interest rate firms face. In any given period, some firms are in the market, the rest not. Firms are allowed to freely enter or exit the market and so the measure of the firms that are in the market, γ, is an endogenous variable. Firms in the market must be matched with a worker in order to produce. A matched pair of firm and worker creates a job.

In any given period, the total measure of matches formed in the labor market is equal to

$$M(\eta_A, \gamma - \eta_E),$$

where η_A is the measure of the unemployed workers (non-employed and actively looking for a job) in the labor market, and η_E is the measure of the workers that are currently employed when the labor market opens, and hence $\gamma - \eta_E$ is the measure of vacant (recruiting) firms in the labor market. Throughout the paper, we assume the matching function is such that

$$0 \leq M(\eta_A, \gamma - \eta_E) < \min\{\eta_A, \gamma - \eta_E\},$$

so there is always a positive measure of workers and firms that are not matched.

A firm that fails to find a match could either exit the market or to operate as a vacant firm in the remainder of the period, waiting for the labor market to open next period. We follow the literature to assume that a vacant firm must incur a fixed cost $c_0 (\geq 0)$ in order to stay open to job applications.

The matched firm and worker Nash bargain over a dynamic employment contract. This dynamic contract specifies a history contingent rule for compensating and terminating the worker. Once they agree on a specific contract, this contract cannot be renegotiated in any future periods.

Production then takes place immediately after a contract is agreed on. In each period, the employed worker produces a random output $\theta \in \{\theta_1, \cdots, \theta_n\}$ with probabilities $\{\pi_1(a), \cdots, \pi_n(a)\}$, where $a \in A \equiv [\underline{a}, \overline{a}]$ is the worker's effort, $\pi_i : A \to (0, 1)$, and $A \subseteq \mathbb{R}_+$ is the set of possible effort levels. For all $a \in A$, let $\overline{\theta}(a) = \sum_i \pi_i(a)\theta_i$. Assume $\overline{\theta}'(\underline{a}) = \infty$.

The model's information structure is the same as that in the standard model of moral hazard. Specifically, the worker's effort is not observable to the firm, but the output he produces is publicly observable and verifiable. Other parameters of the model are common knowledge to all agents in the model.

There is a risk free asset in the model: for each unit of the good invested in this asset, it returns $(1 + r)$ units of consumption next period. To avoid introducing additional information asymmetry, we assume that all investments in this asset are public information and transferable between workers and firms. Workers also have access to a competitive insurance market where one unit of consumption in the current period can be exchanged for $1/\Delta$ units of consumption in the next period conditional on the worker's survival in the next period.

As part of the model's physical environment, we make four assumptions about the contracts that are feasible between the worker and the firm. First, contracts are subject to a non-negativity constraint that requires that compensation to the worker be non-negative. Second, once the worker and the firm agree on a contract, they can commit to not renegotiating the continuations of the contract in all future dates. Third, firms can fully commit to the terms of a long-term contract, whereas workers' commitment to a long-term contract is limited: workers are free to leave an ongoing long-term contract anytime there is a better outside value. Forth, severance payments must be made in lump-sum amounts to the worker immediately upon termination. Once an employment relationship is terminated, no further interactions between the firm and the worker are feasible.

2.3 Equilibrium

In this section, we formulate the economy's stationary equilibrium. We first describe the economy's aggregate state variables. We then describe the optimization problems that workers and firms face, taking the aggregate states as given. Finally, we require that the aggregate states and individual optimization be consistent with each other, and that the firms in the market be making zero profits.

2.3.1 The aggregate states

At the beginning of each period, the state of the economy is characterized by the following aggregate state variables

$$\Sigma = \{(S_A, \mu_A, \eta_A),\ (S_I, \mu_I, \eta_I),\ (X, \mu_E, \eta_E), \gamma\}.$$

Here, γ is the measure of firms in the market. The scalar $\eta_A \in (1 - \Delta, 1)$ is the measure of the non-employed workers who are actively looking for employment. These workers are distributed over the set $S_A \subseteq \mathbb{R}_+$ according to the distribution function $\mu_A : S_A \to [0, 1]$, where S_A is the set of possible amounts of assets these workers hold. The scalar $\eta_I \in (1 - \Delta, 1)$ is the measure of the non-employed workers who do not participate in the labor force. These workers are distributed over the set $S_I \subseteq \mathbb{R}_+$ according to the distribution function $\mu_I : S_I \to [0, 1]$, where S_I is the set of possible amounts of assets these workers hold. The scalar $\eta_E \in (0, 1)$ denotes the the measure of workers that are employed at the beginning of a period. Finally, the employed workers are distributed over

$$X \equiv \left[\frac{u(0) - \phi(\underline{a})}{1 - \beta\Delta}, \frac{u(\infty) - \phi(0)}{1 - \beta\Delta}\right) \equiv [V_{\min}, V_{\max}),$$

the set of all possible expected utilities of the employed workers at the beginning of a period, the distribution function being $\mu_E : X \to [0, 1]$. Clearly, η_A, η_I and η_E must satisfy

$$\eta_E + \eta_A + \eta_I = 1.$$

2.3.2 Optimization

Conditional on Σ, an optimal solution to the firm's and the worker's optimization problems is a duple

$$\sigma \equiv \left\{ \begin{array}{c} \Omega(V), a(V), [c_i(V), V_i(V)]_{i=1}^n, V \in \Phi \\ \overline{U}; U(V), V \in \Phi \\ v(s), s \in \mathbb{R}_+; V_m(s), s \in S_A, V_n(s), s \in \mathbb{R}_+ \end{array} \right\}$$

where the variables in σ are defined through (i)-(v) in the following.

(i) The tuple $\{\Omega(V), a(V), [c_i(V), V_i(V)]_{i=1}^n, V \in \Phi\}$ is the dynamic contract for the currently employed worker. Here we follow Green (1987) and Spear and Srivastava (1987) to use the worker's beginning of period expected utility as a state variable to summarize the worker's history at the firm. The set $\Phi \subseteq X$ is the state space. This is the set of all expected utilities of the employed worker that can be delivered by a (sub-game perfect) feasible and incentive compatible contract. Note that Φ is an endogenous variable of the model. Then, for all $V \in \Phi$, $a(V)$ denotes the worker's recommended effort in the current period, $\Omega(V)$ denotes the set of worker's output realizations in which the worker is retained, and outside which the worker is terminated. Finally, $c_i(V)$ and $V_i(V)$ are, respectively, the worker's current compensation (consumption) and next period utility if his current output is θ_i.

(ii) For all $V \in \Phi$, $U(V) \in \mathbb{R}$ denotes the value of a firm who currently employs a worker with expected utility V. $\overline{U} \in \mathbb{R}$ is the value of a vacant firm: a firm that is free to hire a new worker at the beginning of a period, before the market opens.

(iii) The set $S_A \subseteq \mathbb{R}_+$ denotes the set of assets of the non-employed workers who choose to participate in the current labor market, and the set $S_I = \mathbb{R}_+/S_A$ denotes the set of assets of the non-employed workers who do not participate in the current labor market. Note that a non-employed worker could choose to stay out of the labor market for a number of periods and then re-enter. The scalars η_A and η_I denote, respectively, the numbers of the non-employed workers that belong to the sets S_A and S_I, respectively. We have $\eta_A + \eta_I = \eta_N$. Finally, $\mu_A : S_A \to [0, 1]$ and $\mu_I : S_I \to [0, 1]$, respectively, are the distributions of the non-employed workers who are in the labor market and those who are not.

(iv) For any given $s \in \mathbb{R}_+$, $v(s) \in \mathbb{R}$ denotes the maximized value of the beginning-of-period (before the labor market opens) expected utility of an non-employed worker with assets s; $V_n(s)$ denotes the ex-post expected utility of this worker conditional on his not being matched with a firm (either he chose not to participate in the market ($s \in S_I$), or he went to the market ($s \in S_A$) but failed to find a match); Finally, $V_m(s)$ denotes the bargained expected utility of the worker conditional on (a) he chose to go to the market (*i.e.*, $s \in S_A$) and (b) he is matched with a firm.

Let λ denote the fraction of the vacant firms to obtain a match in the period:

$$\lambda = \frac{M(\eta_A, \gamma - \eta_E)}{\gamma - \eta_E}. \tag{98}$$

Let ρ denote the fraction of unemployed workers to transition to employment (ratio of hiring out of the pool of the unemployed):

$$\rho = \frac{M(\eta_A, \gamma - \eta_E)}{\eta_A}. \tag{99}$$

The restriction we put on the matching function M in Section 2 ensures $0 < \lambda, \rho < 1$.

Definition 2.1 *We say that σ is an optimal solution to the firm and the worker's optimization problem, conditional upon a given set of the market's states Σ and the implied λ and ρ, if it satisfies the following conditions (I) to (IV).*
Condition (I)

$$\overline{U} = \lambda \int_{S_A} (U(V_m(s)) + s) d\mu_A(s) + (1 - \lambda)\frac{1}{1+r}\overline{U} - c_0 \tag{100}$$

Condition (II) *For all $V \in \Phi$,*

$$U(V) = \max_{\{\Omega, a, c_i, V_i\}} \sum_{i \notin \Omega} \pi_i(a) \left[\theta_i - c_i + \frac{\Delta}{1+r}\left(\overline{U} - v^{-1}(V_i)\right)\right] + \sum_{i \in \Omega} \pi_i(a) \left[\theta_i - c_i + \frac{\Delta}{1+r} U(V_i)\right] + \frac{1-\Delta}{1+r}\overline{U} \tag{101}$$

subject to (102)-(106) where

$$V = \sum_{i=1}^{n} \pi_i(a)[u(c_i) + \beta \Delta V_i] - \phi(a) \tag{102}$$

$$a = \arg\max_{a' \in A} \left(\sum_{i=1}^{n} \pi_i(a')[u(c_i) + \beta \Delta V_i] - \phi(a') \right) \tag{103}$$

$$\Omega \subseteq \Theta, \tag{104}$$

$$c_i \geq 0, \ \forall i \tag{105}$$

$$V_i \in \Phi, \ \forall i \in \Omega \tag{106}$$

$$V_i \in v(\mathbb{R}_+),\ \forall i \notin \Omega \tag{107}$$

$$V_i \geq v(0),\ \forall i \tag{108}$$

where the function $v : \mathbb{R}_+ \to \mathbb{R}$, its inverse v^{-1}, which is to be shown to exist later in the paper, and the value of $v(0)$, will be given in Condition (IV);

Condition (III) *The set Φ of all expected utilities for an employed worker that can be generated by a feasible and incentive compatible contract is the largest self-generating set of the mapping $B : 2^X \to 2^X$ defined by: $\forall \Phi' \subseteq X$,*

$$B(\Phi') \equiv \{V \in X | \exists \{\Omega, a, c_i, V_i\}\ s.t.\ (102)-(105), (107), (108),\ \text{and } V_i \in \Phi'\ \forall i \in \Omega\}. \tag{109}$$

Condition (IV) *The non-employed-worker's problem about whether to enter the labor market and the related values are described by*

$$S_A \equiv \left\{ s \in \mathbb{R}_+ : \exists V \in \Phi, V \geq V_n(s)\ \textit{such that}\ U(V) + s \geq \frac{1}{1+r}\overline{U} \right\}, \tag{110}$$

$$S_I \equiv \mathbb{R}_+ \diagdown S_A \tag{111}$$

where

$$V_n(s) = \max_{0 \leq c \leq s} \left\{ u(c) - \phi(0) + \beta \Delta v \left(\frac{1+r}{\Delta}(s-c) \right) \right\}\ \forall s \in [0, +\infty), \tag{112}$$

$$V_m(s) = \arg \max_{V \in \Phi, V \geq V_n(s), U(V)+s-\frac{1}{1+r}\overline{U} \geq 0} \left(U(V) + s - \frac{1}{1+r}\overline{U} \right)^{\omega} (V - V_n(s))^{1-\omega},\ \forall s \in S_A \tag{113}$$

$$\forall s \in \mathbb{R}_+,\ v(s) = \begin{cases} \rho V_m(s) + (1-\rho) V_n(s)\ , & \textit{if } s \in S_A \\ V_n(s) & , \textit{if } s \in S_I \end{cases}. \tag{114}$$

Conditions (I)-(IV) formulate a set of Bellman equations for the values of the firms and the workers, along with the optimal strategies.

Condition (I) gives the value of a vacant firm at the beginning of a period. With probability λ this vacant firm is matched with an unemployed worker with assets s who is drawn randomly from the distribution μ_A. Once matched, the worker gives his assets s to the firm, and the firm gives the worker an employment contract that promises the worker expected utility $V_m(s)$. This $V_m(s)$ is the solution to the Nash bargaining problem to be formulated in equation (113).

Implicitly in equation (4) is the assumption that assets are freely transferable between the worker and the firm. Suppose assets are not freely transferable. Suppose for example assets could not be transferred at all between the worker and the firm. Then an additional state variable will be needed for recursively formulating the dynamic contract. We leave this possibility for future work.

Condition (II) gives the optimal dynamic contract between an employed worker and the firm, along with the value of the firm as a function of the worker's expected utility. In equation (5), $v^{-1}(V_i)$ is the cost to the firm of letting the worker leave the firm with a promised utility equal to V_i. Here v^{-1} is the inverse of the worker's value function v which, in turn, is defined in equation (18). That is, in order to guarantee that the worker obtains a level of expected utility equal to V_i, the firm must make a severance payment to the worker in the amount $v^{-1}(V_i)$. Note that at this stage, it is not clear whether the inverse function v^{-1}, and the function V_m in equation (4) are well defined. In the next section, we will show that the function v is indeed well defined, continuous, and strictly increasing over its domain $\mathbb{R}_+$, and hence its inverse exists and is monotonic. We will also show that $V_m(s)$ is well defined for each $s \in S_A$.

Since each worker dies with probability $(1 - \Delta)$, the firm faces in each period a constant probability of $(1 - \Delta)$ to become vacant next period.

Constraint (6) is promise-keeping. Constraint (7) is incentive compatibility. Equation (8) says that the contract can terminate in any chosen subset of the worker's outputs. Equation (9) requires the worker's compensation to be non-negative: the limited liability constraint.

Constraints (106) and (107) require that the expected utility V_i promised to the worker be feasible. Specifically, if the worker is retained, then the promised utility must be achievable by a sub-game perfect feasible and incentive compatible contract; if the worker is terminated, then the expected utility the worker receives must be supportable by a feasible severance payment s.

Equation (12) is a self-enforcing constraint. Under this constraint, the worker will not have an incentive to leave the contract in all ex post states of the world. This constraint is not needed if we assume full commitment.

Condition (III) provides a Bellman equation for the state space of the dynamic contract. This follows Abreu, Pearce and Stachetti (1990) and Wang (1997).

Equation (14) defines the set of non-employed workers that are in the market, S_A. The condition it imposes on all $s \in S_A$ say that, in order for a non-employed worker to be willing to participate in the labor force, there must exist a feasible and incentive compatible contract to make the worker and the firm both better off should they form a match.

Equation (16) describes the optimization problem for the non-employed worker who is not matched with a firm, either he was in the market but failed to form a match, or he chose not to participate in the labor market. The problem for this worker, which is the same as that for the consumer in a typical growth model, is one of finding the optimal consumption and saving scheme.

Equation (17) lays out the problem of Nash bargaining between a worker and a firm who are matched. The parameter $\omega \in (0,1)$ is the exogenously given bargaining weight for the firm. Since in each period each firm and each worker can find at most one match, $\overline{U}/(1+r)$ is the firm's reservation utility, and $V_n(s)$ is the worker's. The bargaining game here thus involves choosing a level of the expected utility V in the set Φ of attainable expected utilities to give to the worker, and this expected utility exceeds the worker's reservation utility, makes the firm better off than its reservation utility, and maximizes the Nash product of surpluses.

Note that implicit in Equation (17) is the assumption that (a) the Nash bargaining problem has a solution and (b) the solution is unique. Proposition 2.2 in the next section will verify that this assumption is satisfied.

Finally, equation (18) describes the non-employed worker's decision about whether or not to participate in the current labor market. Note that since the worker does not incur any costs being in the labor market, we make the assumption that workers choose to participate in the labor market if and only if they could with a match that offers an expected utility that is higher than $V_n(s)$. Workers who have a zero probability to be hired will voluntarily stay out of the labor market.

2.3.3 Equilibrium

Definition 2.2 *A stationary equilibrium of the model is a tuple $\{\Sigma, \lambda, \rho, \sigma\}$ that satisfies the following conditions:*

(i) λ and ρ are given by (98) and (99).

(ii) Conditional on Σ, ρ and λ, σ solves the worker and the firm's optimization problems that are defined by Conditions (I)-(IV) in Definition 1.

(iii) Σ is generated by σ and is stationary.

(iv) Free entry of firms into the market ensures

$$\overline{U} = 0.$$

2.4 Analysis

In this section, we analyze the Bellman equations in Definition 1 that jointly characterize the worker and the firm's optimization problems. We begin with a set of useful observations. Observe first that

$$V_m(s) \geq v(s) \geq V_n(s), \ \forall s \in S_A. \tag{115}$$

This holds because the definitions of S_A and $V_m(s)$ imply $V_m(s) \geq V_n(s)$ and that $v(s)$ is a convex combination of $V_m(s)$ and $V_n(s)$ for all $s \in S_A$. Notice next that

$$V_n(0) = u(0) - \phi(\underline{a}) + \beta\Delta v(0).$$

Notice also that

$$v(0) \geq V_{\min}.$$

This holds because the non-employed worker with $s = 0$ can always choose to stay out of the labor market permanently to obtain V_{min}. Notice therefore

$$v(0) \geq V_n(0) = u(0) - \phi(\underline{a}) + \beta\Delta v(0) \geq V_{min}. \tag{116}$$

Assumption 1 $u(\infty) - u(0) \geq \beta\Delta[\phi(\underline{a}) - \phi(0)]$.

This assumption is not difficult to satisfy. A sufficient condition for it to hold is $u(\infty) - \phi(\underline{a}) \geq u(0) - \phi(0)$. That is, the worker is better off working and having an infinite amount of consumption than not working and not consuming. But we need Assumption 1 to show that the set ϕ is an interval.

Proposition 2.1 $\Phi = \left[V_n(0),\ \frac{u(\infty)-\phi(\underline{a})}{1-\beta\Delta}\right)$.

Remember $V_n(0) = u(0) - \phi(\underline{a}) + \beta\Delta v(0)$ is the expected utility of a worker who is not matched with a firm and has zero assets.

Assumption 2 The value function $U : \Phi \to \mathbb{R}$ is continuous and concave.

This assumption is reasonable, for the continuity and concavity of U could always be obtained through randomization over employment contracts if necessary. See Athey and Bagwell (2001).

Notice that $U(V) \to -\infty$ as $V \to [u(\infty) - \phi(0)]/(1 - \beta\Delta)$. This holds because, independent of the contract used, the expected profits of a firm are bounded from above while the cost of delivering V to the worker goes to infinity as V goes to $[u(\infty) - \phi(0)]/(1 - \beta\Delta)$. With this and Assumption 1, let

$$V^* = \max\left\{V' :\ V' \in \arg\max_{V\in\Phi} U(V)\right\}.$$

This V^* exists, is unique, and has the following interpretation: If the firm is free to offer any expected utility from Φ to a newly hired worker with assets $s = 0$, V^* is the starting expected utility to promise to the worker at which the firm can achieve its maximum value, $U(V^*)$.

Notice that V^* may not be equal to the bargained expected utility $V_m(0)$, although it does always hold that $V^* \leq V_m(0)$. Suppose $V^* > V_m(0)$. Then by letting $V_m(0) =$

V^* the worker is strictly better off and the firm is weakly better off. Notice next that since V^* is taken from the set Φ, and from Proposition 2.1 $V_n(0)$ is the minimum element in Φ, it holds automatically that $V^* \geq V_n(0)$. Moreover, with the assumption that $\overline{\theta}'(0) = \infty$, this inequality must hold strictly, that is,

$$V^* > V_n(0).$$

In other words, the firm would start a new worker that has zero assets with an expected utility that is strictly greater than his reservation expected utility. To see this, suppose $V_* = V_n(0)$. Then the optimal contract entails that the worker's effort is the minimum $\underline{a}$ and his compensation is 0, for otherwise his expected utility would strictly exceed $V_n(0)$. But this is not optimal given $\phi'(\underline{a}) < \overline{\theta}'(0) = \infty$.

We now proceed to provide an analysis of the non-employed worker's problem that is defined by Condition (V) in Definition 1. If analysis will be partial, in the sense that we will take the vacant firm's value $\overline{U}$ and the non-vacant firm's value function $U(\cdot)$ as given and seek to characterize the value functions $v(\cdot)$, $V_n(\cdot)$, and $V_m(\cdot)$.

Notice that the function $v(\cdot)$ plays a central role in defining the firm and worker's optimization problems. First, $v(\cdot)$ provides a link between the firm's optimization problem [equations (4)-(13)] and the worker's problem [equations (14)-(18)]. Second, as we will show, if $v(\cdot)$ is well defined and continuous, then the worker's other value functions $V_n(\cdot)$ and $V_m(\cdot)$ are also well defined and continuous. Given this, the strategy of our analysis is to formulate the function $v(\cdot)$ as a fixed point of a contraction mapping on a space of bounded and continuous function, and then use the contraction mapping theorem to obtain that $v(\cdot)$ is uniquely defined and continuous[43].

Assumption 3 There exists a feasible and incentive compatible one-period contract σ_0 that offers the worker expected utility $V_0 \geq u(0) - \phi(0)$ and the firm expected profit $\Pi(V_0) > 0$.

Proposition 2.2 *Given the vacant firm's value $\overline{U}$ and non-vacant firm's value function $U(V), V \in \Phi$, the following holds for the non-employed worker. (i) The non-employed worker's value function $v(\cdot)$ is well defined, continuous, and strictly increasing in $\mathbb{R}_+$. (ii) The non-matched worker's value function $V_n(\cdot)$ is well defined, continuous, and strictly increasing on $\mathbb{R}_+$; (iii) The matched worker's value function V_m is well defined, continuous, and weakly increasing on S_A;*

[43]A full analysis of the firm and worker's optimization problem (as defined in Definition 1) would require characterizing the value functions $\overline{U}$, $U(\cdot)$ and $v(\cdot)$ simultaneously in a unified fixed point argument. A difficulty is that the function $U(\cdot)$ is not bounded so the contraction mapping theorem could not be applied for the proof of existence.

We also have

$$v(\mathbb{R}_+) = [v(0), V_{max}).$$

With this, and given (116), we have

$$[v(0), V_{max}) \subseteq \Phi,$$

and we can rewrite constraints (106)-(108) as

$$V_i \in [v(0), V_{max}).$$

This leads to

Proposition 2.3 *With the optimal contract,* $i \in \Omega$ *if and if* $U(V_i) > \overline{U} - v^{-1}(V_i)$

This result is intuitive, it says that the worker is retained if the value of retention is greater than the value of termination. Notice that since in equilibrium $\overline{U} = 0$, the above proposition permits the firm to retain a worker that has a negative value to the firm, as long as the value of terminating him is even lower.

Proposition 2.4 *Suppose a newly-terminated worker with expected utility* V *goes back to the labor market immediately* [*i.e.,* $v^{-1}(V) \in S_A$]. *Then either* $\overline{U} > 0$*, or there exists* $V' \in \Phi$ *such that* $V' > V$ *and* $U(V') > U(V)$*.*

Given that $U(V)$ is concave, in equilibrium with $\overline{U} = 0$, in order for a worker to go back to the labor market immediately after termination, his expected utility must be sufficiently low, lower than V^*. In other words, a newly terminated worker is unemployed if he is terminated from the left hand side of the firm's value function. Worker who are terminated from the right hand side of the firm's value function will stay out of the labor force for at least one period.

Consider the set of the promised expected utilities of the worker upon which the worker is retained. Then for each V in this set, consider the amount of assets $v^{-1}(V)$ the worker would receive if the worker were terminated. Let

$$\widehat{\Omega} \equiv \{v^{-1}(V) | V \in [v(0), V_{max}),\ U(V) > \overline{U} - v^{-1}(V)\}.$$

Next, let

$$\widehat{\Omega}_A \equiv \{v^{-1}(V) | V \in [v(0), V_*),\ U(V) < \overline{U} - v^{-1}(V)\},$$

$$\widehat{\Omega}_I \equiv \{v^{-1}(V) | V \in [V_*, V_{max}),\ U(V) < \overline{U} - v^{-1}(V)\}.$$

Each element in $\widehat{\Omega}_A$ ($\widehat{\Omega}_I$) is a level of assets that corresponds to a level of utility V at which the worker is terminated from the left (right) hand side of the firm's value function. The following proposition establishes the connection between the firm's decision about when to terminate the worker and the worker's decision about when to enter the labor market.

Proposition 2.5 *In equilibrium it holds that* $S_A = \widehat{\Omega}_A \bigcup \widehat{\Omega}$ *and* $S_I = \widehat{\Omega}_I$.

Proposition 2.6 *There exists* $\overline{s} > 0$ *such that* $[0, \overline{s}] \subseteq S_A$. *Moreover, if* $\underline{a} = 0$, *then* $[0, v^{-1}(V^*)] \subseteq S_A$.

So a non-employer worker must be unemployed if he is sufficiently poor. This is intuitive, for his value of staying out of the labor force is lower given his small s.

Assumption 3 There exists $\overline{V}$ such that the following inequality holds for all $V \geq \overline{V}$.

$$\overline{\theta} - (1 - \beta\Delta)\overline{U} \leq u^{-1}[(1 - \beta\Delta)V + \phi(\underline{a})] - u^{-1}[(1 - \beta\Delta)V].$$

Assumption 3 is easily satisfied. Notice that since $u(\cdot)$ is concave, $u^{-1}(\cdot)$ is convex, and $u^{-1}[(1 - \beta\Delta)V + \phi(\underline{a})] - u^{-1}[(1 - \beta\Delta)V]$ is increasing in V.

Proposition 2.7 *Under Assumption 3, the set* $\widehat{\Omega}$ *is bounded. Or equivalently, there exists* $\overline{V} > v(0)$ *such that for all* $V \geq \overline{V}$, $U(V) < \overline{U} - v^{-1}(V)$.

This proposition says that a worker cannot be employed with a level of expected utility that is sufficiently high: he would have been terminated. In fact, by Proposition 2.8, a non-employed worker with a sufficiently high V will choose not to be in the labor market.

Among workers that choose not to participate in the labor force currently, there are those who have chosen to quit the labor force permanently, and those who are just temporarily out of the labor force. Our next proposition shows that non-employed workers who are rich enough, with a large enough s, will choose to retire permanently from the labor market.

Proposition 2.8 *Assume* $\beta = \frac{1}{1+r}$. *Then there exists* $s^* < \infty$ *such that the non-employed worker will choose to stay out of the labor force permanently if* $s \geq s^*$.

We therefore have the following expression for a part of the function $v(\cdot)$:

$$v(s) = \frac{u(\frac{1+r}{\Delta}s) - \phi(0)}{1 - \beta\Delta}, \ \forall s \geq s^*.$$

2.5 Quantitative Analysis

In this section, we calibrate our model to the U.S. data, analyze it numerically, and show that our model could do a better job accounting for the observed wage dispersion than standard search/matching models.

2.5.1 Parameterization and Calibration

We set the time period to be one month. We set the discount rate to be $r = 0.00417$ to obtain an annual interest rate of 5%. We then set the worker's discount factor to be $\beta = 1/(1+r)$. We set $\Delta = 0.99815$ so the worker's expected lifetime is 45 years.

We set the worker's utility function to be

$$u(c) - \phi(a) = \log(\rho_0 + \rho_1 c) - a^2,$$

where ρ_0 is normalized to 1 and $\rho_1 > 0$.

We set $n = 2$ so output can be low (θ_1) or high (θ_2). We assume

$$\pi_1(a) = \exp(-\psi a),\ \pi_2(a) = 1 - \exp(-\psi a),\ \forall a \geq 0,$$

where $\psi > 0$. We follow the literature to assume a Cobb-Douglas matching function so that

$$M(\eta_A, \gamma - \eta_E) = \alpha_0 {\eta_A}^{\alpha} (\gamma - \eta_E)^{1-\alpha}$$

The above parameterization leaves us with the following parameters for the calibration of the model:

$$\theta_1,\ \theta_2,\ \rho_1, \psi, \alpha_0, c_0, \gamma.$$

We target a measure of unemployed workers equal to 0.0342, a measure of employed workers equal to 0.6336, and a measure of those not in the labor force equal to 0.3320. These values are derived from The Current Population Survey (CPS) which provides monthly time series data on employment, unemployment and not-in-the-labor-force, for the period between January 1994 and December 2003. These target measures imply an unemployment rate of 5.12%, and a labor force participation rate of 66.78%.

We target a job finding probability of 28.3%, following Fallick and Fleischman (2004). We follow the literature to set $\alpha = 0.6$. The literature reports a value of α between 0.5 and 0.7 (Blanchard and Diamond (1989), Petrongolo and Pissarides (2001)).

Davis, Faberman and Haltiwanger (2007) reports a job opening rate of 3.4% for the period from December 2000 to January 2005[44]. Using this information, we choose the value of α_0 to generate a job finding probability (fraction of the unemployed to flow into employment) of 28.3%:

$$\alpha_0 \left(\frac{\gamma - \eta_E}{\eta_A}\right)^{1-0.6} = \alpha_0 \left(\frac{0.034 * 0.6336}{0.0342}\right)^{1-0.6} = 0.283$$

which gives us $\alpha_0 = 0.3405$. In addition, given that

$$\text{job opening rate} = \frac{\gamma - \text{employment}}{\text{employment}}$$

we obtain $\gamma = 1.034 \times 0.6338 = 0.6551$.

We follow Shimer (2005) to set $\omega = 0.4$ (Hosios 1990). We could alternatively set $\omega = 0.5$ without significantly change the calibration outcome.

We are now left with five free parameters $\theta_1, \theta_2, \psi, \rho_1, c_0$, and we choose their values to target 6 (essentially 5) measures of the U.S. data: the measures of employment (E), unemployment (U), non-participation (N); the rate of flow from employment to unemployment, the job finding probability (rate of flow from unemployment to employment), and the job opening rate (vacancies as a fraction of employment).

The following table gives the values of the parameters chosen.

Parameter	Value
θ_1	−0.5000
θ_2	2.5000
ψ	0.6386
ρ_1	1.2771
c_0	0.0096

The following table compares the calibrated model with data.

Variable	Data	Model
fraction of employment	0.6336	0.6317
fraction of unemployment	0.0342	0.0350
fraction of not in the labor force	0.3320	0.3333
job opening rate	3.4%	3.7%
E to U probability	1.3%	1.26%
U to E probability	28.3%	29.1%

The model does a good job matching the targets. Note that conditional on em-

[44]Their measure is based on the Job Openings and Labor Turnover Survey (JOLTS).

ployment, which is an independent target to match, the job opening rate essentially measures the stock of vacancies in the economy.

The U.S. data shows a large flow from unemployment to not-in-the-labor-force, reflecting the movements of discouraged workers, and the movements from unemployment to education. Our model lacks a channel for the flow from unemployment to NLF.

The U.S. data also shows a significant flow of workers from not-in-the-labor-force to employment. This reflects the fact that, in practice, firms search not only among workers that are unemployed, but also among workers that are not in the labor force. This mechanism is missing in our model. In the model, workers must be actively looking for jobs before being matched with a firm.

Finally, notice that the flows from employment to unemployment and from employment to not-in-the-labor-force are much smaller in the model than in the data. These are not surprising. In the data, a large fraction of the transitions from employment to not in the labor force are due to life-cycle reasons, or younger workers quitting the labor force to obtain higher education. These are not in our model. In the data, the flow from not in the labor force to unemployment reflects perhaps the movements of the previously discouraged workers and the young workers who enter the labor market after finishing education.

2.5.2 Equilibrium

Figure 1 depicts the firm's net gains from retaining (rather than terminating) the worker as a function of the worker's expected utility. In order to deliver a given level of expected utility V to the worker, the firm's net profits are $U(V)$ if it retains the worker and $\overline{U} - v^{-1}(V)$ if it terminates the worker, and in equilibrium $\overline{U} = 0$. The value of the difference is shown in Figure 1. Obviously, termination is optimal if and only the value of V is sufficiently small or sufficiently large.

Figure 2 depicts the law of motion for the employed worker's expected utility as a function of his current output. The worker's expected utility is higher (lower) next period if his current is higher (lower) this period.

Figure 4 shows the (deterministic) law of motion for the worker's assets: $s_{t+1} - s_t$ as a function of s_t. There is critical asset level above which the non-employed worker chooses not to enter the labor market. For sufficiently high asset levels, there is not a non-negative Nash surplus between the worker and the firm.

The stationary distributions of employed workers and non-employed workers (unemployed plus not-in-in-the-labor-force) are shown in Figures 5 and 6, respectively. There is clearly a significant amount of welfare dispersion among the employed workers.

At each point in time, looking forward each employed worker faces a stochastic number of periods over which to remain employed. Figure 8 depicts the distribution of the duration of the current job for a worker with four different level of starting expected utilities. Obviously, the worker who has an expected utility that is neither too low nor too higher will longer on this current job on average.

In equilibrium, a worker who leaves his job with an expected utility above the upper bound of the retention interval will not go back to the labor market immediately, a worker who leaves his job with an expected utility below the lower bound of the retention interval will go back to labor market right away. Hence, the former consists of the employment to not-in-the-labor-force transition, and the latter consists of the employment to unemployment transition.

Furthermore, each worker is born without any saving. As a result, his first job is characterized by a contract delivering relatively low expected utility. It takes time for him to establish a good record, and in turn be promised a relatively high expected utility.

Figure 9 is based on simulation and shows that the probability to transition from employment to unemployment decreases with the age of the worker, while the probability to transition from employment to not-in-the-labor-force is increasing with the age of the worker. These are consistent with findings in Nagypal (2005)[45].

2.5.3 Wage Dispersion

Hornstein, Krusell and Violante (2006) show that standard search matching models can generate only a very small, 3.6%, differential between the average wage and the lowest wage paid in the labor market, whereas the observed Mm ratio–the ratio between the average wage and lowest wage paid– is at least twenty times larger than what the model observes. Hornstein, Krusell and Violante further show that the extensions to the standard search and matching models can only modestly improve their performance on accounting for the observed Mm ratio. As HKV argue, the logic of the search/matching model implies that a higher wage dispersion is associated with longer unemployment durations or a smaller probability of finding employment for the unemployed. Given that unemployment durations are typically short in the data, wage dispersion cannot be large in the model[46].

This logic of the search/matching model does not apply in our model. In our model, wage dispersion is driven by the provision of intertemporal incentives and

[45]In Nagypal (2005), the probability to transition from employment to not-in-the-labor-force for younger workers is unusually high which might be explained by the higher education admission.

[46]As the paper explains, "The short unemployment durations, as in the U.S. data, reveal that agents in the model do not find it worthwhile to wait because frictional wage inequality is tiny. The message of search theory is that "good things come to those who wait", so if the wait is short, it must be that good things are not likely to happen." (page 9.)

risk sharing. Wages of homogenous workers who start with the same initial expected utility fan out over time as their outputs follow a stochastic process. In our model, workers who produce a high output not only receive a higher wage in the current period, but also will see their future utilities and wages increased. Likewise, and workers that produce a low outputs will receive lower wages in the current period and in the future[47].

Our model is capable of generating much larger wage dispersions than standard search and matching models do. In the version of the model that is calibrated to the U.S. data, the computed average wage is 0.4071, and the lowest wage paid is 0.0166, and the Mm ratio is 24.5, similar to what Hornstein, Krusell and Violante observe in the U.S. data.

Suppose we use the average wage of the workers in the lowest wage percentile as the minimum wage in the calculation, then the computed Mm ratio is 13.89. Even if we use the average wage of the workers in the 5th wage percentile as the minimum wage in the calculation, the computed Mm ratio is 5.32, much larger than what the search/matching models permit. Note that our model generates the same job finding probability for the unemployed, and hence the same average unemployment duration, as the calibrated search/matching models do.

2.6 Conclusion

In this paper, we have studied an equilibrium model of the labor market which modifies the Mortensen-Pissarides framework by taking a dynamic contract approach to jobs and job separation. The dynamic contract is motivated by a standard information friction: moral hazard. Optimal terminations of dynamic contracts generate equilibrium worker flows from employment to unemployment and to not-in-the-labor-force. Matching and bargaining bring unemployed workers to employment. As in the data, in the model average wages increase with worker tenure, and on average workers who have stayed longer with the firm face lower layoff probabilities. Our model offers an important advantage over standard search and matching models: we have shown quantitatively that our model generates wage dispersions that are similar to those observed in the data while standard search-matching models cannot.

Our model has a number of possible extensions among which perhaps the most important and challenging is to add aggregate uncertainties to our currently stationary environment. As discussed in the introduction of the paper, existing search-matching models have not been able to account for the observed pattern of wage dynamics over the business cycle. Our model offers a natural and promising alternative, given

[47]The effect of the dynamic contracting on distribution was first discussed in Green (1987) and Atkeson and Lucas (1991).

that the structure of compensation is designed to achieve efficient intertemporal risk sharing in our model.

2.7 Appendix

2.7.1 Proof of Proposition 2.1

Let Φ' denote the set on the right side of the equation we need to show to hold.

We first show $\Phi \subseteq \Phi'$. There does not exist $V \in \Phi$ such that $V < u(0) - \phi(\underline{a}) + \beta\Delta v(0)$, since the worker can always choose to exert the lowest effort $\underline{a}$ regardless of the contract offered, and he is guaranteed expected utility $v(0)$ next period. It is also obvious that there exists no $V \in \Phi$ such that $V \geq u(\infty) - \phi(\underline{a}) + \beta\Delta V_{\max}$.

Next, we show Φ' is self-generating and hence $\Phi' \subseteq \Phi$.

For all $V \in \Phi'$ with $V < (1+\beta\Delta)[u(\infty) - \phi(\underline{a})] + (\beta\Delta)^2 V_{\max} (< u(\infty) - \phi(\underline{a}) + \beta\Delta V_{max})$, let

$$\Omega(V) = \Theta,\ a(V) = \underline{a},\ c_i(V) = x(V),\ \text{and } V_i(V) = y(V)$$

where $x(V) \in \mathbb{R}_+$ and $y(V) \in [v(0), u(\infty) - \phi(\underline{a}) + \beta\Delta V_{\max}) \subseteq \Phi'$ are chosen to satisfy

$$V = u(x(V)) - \phi(\underline{a}) + \beta\Delta y(V).$$

Obviously, such $(\Omega(V), a(V), c_i(V), V_i(V))$ is feasible, satisfies (6)-(9), (11), (12), and $V_i(V) \in \Omega(V)\ \forall i \in \Omega(V)$, and therefore generates V.

Next, for all $V \in \Phi'$ with $V \geq (1+\beta\Delta)[u(\infty) - \phi(\underline{a})] + (\beta\Delta)^2 V_{\max}$, let

$$\Omega(V) = \emptyset,\ a(V) = \underline{a},\ c_i(V) = x(V) \text{ and } V_i(V) = v(s(V))$$

where $x(V), s(V) \in \mathbb{R}_+$ are chosen to satisfy

$$V = u(x(V)) - \phi(\underline{a}) + \beta\Delta v(s(V)).$$

We now show that this is feasible to do. Observe first that

$$u(0) - \phi(\underline{a}) + \beta\Delta V_{\max} < (1+\beta\Delta)[u(\infty) - \phi(\underline{a})] + (\beta\Delta)^2 V_{\max}.$$

This is directly implied by Assumption 1. Let

$$\epsilon \equiv \left[(1+\beta\Delta)[u(\infty) - \phi(\underline{a})] + (\beta\Delta)^2 V_{\max}\right] - [u(0) - \phi(\underline{a}) + \beta\Delta V_{\max}] > 0.$$

Now for a fixed $V \in [(1+\beta\Delta)[u(\infty) - \phi(\underline{a})] + (\beta\Delta)^2 V_{\max},\ u(\infty) - \phi(\underline{a}) + \beta\Delta V_{max}]$, let $s(V) = V_{\max} - 0.5 * \epsilon$. Since $u(\cdot)$ is continuous, we can then choose $x(V) \geq 0$ so that the above equation is satisfied. Last, it is easy to see that the so chosen $(\Omega(V), a(V), c_i(V), V_i(V))$ satisfies (6)-(9), (11), (12), and $V_i(V) \in \Omega(V)\ \forall i \in \Omega(V)$, and therefore generates V. The proposition is proved.

2.7.2 Proof of Proposition 2.2

For any $s \in S_A$, if $v(s)$ is well defined, then the solution to the bargaining problem (113) exists and is unique.

We assume that he value function $U : \Phi \longrightarrow \mathbb{R}$ is continuous and concave. This assumption is reasonable, for the continuity and concavity of U could always be obtained through randomization over employment contracts if necessary. See Athey and Bagwell (2001).

Let $s \in S_A$. We show that the solution to the following optimization problem exists and is unique:

$$\max O(V) \; s.t. \; V \in \Phi, \; V \geq V_n(s), \; U(V) + s - \beta \overline{U} \geq 0$$

where

$$O(V) \equiv (U(V) + s - \beta \overline{U})^{\omega}(V - V_n(s))^{1-\omega}.$$

We first prove existence. Notice first that the constraint $V \in \Phi$ is not binding. To show this, notice $V_n(s) \geq V_n(0)$, then use the observation

$$V_n(0) = u(0) - \phi(0) + \beta \Delta v(0).$$

Notice next that since $U(V) \rightarrow -\infty$ as $V \rightarrow V_{max}$, the constraint $V \geq V_n(s)$ can be replaced by $V_n(s) \leq V \leq M$ for some sufficiently large M.

Since $U(V)$ is continuous, we have that the constraint set, which can now be written as $\{V \in \mathbb{R} : \; V_n(s) \leq V \leq M, \; U(V) + s - \beta \overline{U} \geq 0\}$, is closed and bounded, and hence compact. Since the objective function $O(V)$ is continuous, a solution exists.

We now prove uniqueness. This takes 5 steps.

(i) Notice first that V is not optimal if $V < V^*$, where V^* is defined in equation (**??**). To show this, suppose $V \in V^*$. Then $V' = V + \epsilon$ could make both the firm and the worker strictly better off, for a positive but sufficiently small ϵ; a contradiction.

(ii) Notice next that since $U(V)$ is concave by Assumption 2, $U(V)$ and hence $U(V) + s - \beta \overline{U}$ are strictly decreasing over $[V^*, v(\infty)/(1 - \beta \Delta))$. Since $s \in S_A$ requires $U(V^*) + s - \beta \overline{U} \geq 0$, the equation $U(V) + s - \beta \overline{U} = 0$ has a unique solution. Denote it $\overline{V}(s)$. This allows us to rewrite the constraint $U(V) + s \geq \beta \overline{U}$ as

$$V \leq \overline{V}(s).$$

(iii) Notice next then

$$V \in \{V' \in \mathbb{R} : \; V_n(s) \leq V' \leq M, \; U(V') + s - \beta\overline{U} \geq 0\}$$

if and only if

$$V_n(s) \leq V \leq \overline{V}(s),$$

where it must hold that $V_n(s) \leq \overline{V}(s)$ since the feasibility set cannot be empty. Now suppose $V_n(s) = \overline{V}(s)$. Then of course there is a unique solution that maximizes $O(V)$. In the following we show that the solution to the bargaining problem is unique also in the case $V_n(s) < \overline{V}(s)$.

(iv) So suppose $V_n(s) < \overline{V}(s)$. Notice first that a solution must satisfy either $V = V_n(s)$ or $V = \overline{V}(s)$, or $O'(V) = 0$.

Notice that $V = V_n(s)$ cannot be optimal, because $V' = V_n(s) + \epsilon$ with ϵ positive but sufficiently small can attain $O(V') > 0 = O(V)$. (Note it doesn't matter whether $V_n(s) \geq V^*$ or otherwise.)

Notice that $V = \overline{V}(s)$ cannot be optimal either, because $V' = V_n(s) - \epsilon$ with ϵ positive but sufficiently small can attain $O(V') > 0 = O(V)$.

Therefore, any solution V must satisfy $O'(V) = 0$, or

$$-U'(V)\frac{V - V_n(s)}{U(V) + s - \beta\overline{U}} = \frac{1-\omega}{\omega}.$$

(v) Observe first that in order to have a solution, it must hold that $U'(V) < 0$, otherwise the left hand side of the equation is non-positive while the right hand side is strictly positive. Thus, we need only consider the set of Vs over which the value function $U(V)$ is strictly decreasing. Given that $U(V)$ is concave, this in turn implies that the left hand side is strictly increasing in V over the set of Vs that could potentially solve the problem. It then follows that there at most one $V = V_m(s)$.

To conclude the proof of the lemma, note that we have proved (ii) under the assumption that the value function U is differentiable. A proof that does not rely on the differentiability of U is available upon request.

Under Assumption 2, $0 \in S_A$.

Suppose $0 \notin S_A$. That is, suppose $0 \in S_I$. Then $v(0) = V_n(0) = V_{min} = \frac{u(0)-\phi(0)}{1-\beta\Delta}$. Now consider the following contract: it is σ_0 for the first period, and then the worker is given $s = 0$ to leave the firm. This contract delivers an expected utility equal to $V_0 + \beta\Delta v(0)$ to the worker. Clearly $V_0 + \beta\Delta v(0) \geq V_n(0)$ and $V_0 + \beta\Delta v(0) \in \Phi$. This contract gives an expected profit equal to $\Pi(V_0) + \beta\overline{U}$ to firm. Now $U(V_0 + \beta\Delta v(0)) \geq \Pi(V_0) + \beta\overline{U} \geq \beta\overline{U}$. So $s = 0 \in S_A$. A contraction.

Suppose the function v is well defined and continuous. Then (i) V_n is a well defined, continuous, and strictly increasing function on $\mathbb{R}_+$; (ii) V_m is well defined, continuous, and increasing on S_A; and (iii) v is strictly increasing on $\mathbb{R}_+$.

Let v be well defined and continuous. Let $s_2 > s_1 \geq 0$.

(i) That $V_n(s)$ is well defined and continuous is because the objective function is continuous the constraint correspondence is compact. Use then the theorem of the maximum. To show that $V_n(s)$ is strictly increasing in s, notice that with s_2, the worker can always choose to have strictly more consumption in the current period while setting his future assets equal to that with s_1.

(ii) We show that the function $V_m(s)$ is also continuous. This is the case because: (a) The objective function in (113) is continuous in V. (b) Given $U(V) \to -\infty$ as $V \to V_{max}$, there is some $M > 0$ sufficiently large such that for each $s \in S_A$, the constraint $V \geq V_n(s)$ can be replaced by $V_n(s) \leq V \leq M$. This implies a constraint correspondence that is compact valued and continuous. (c) Apply the theorem of the maximum.

We next show that V_m is an increasing function. Observe first that given $U'(V) < 0$ at the optimal V, and $V_n(s)$ is increasing in s, the left hand side of (**??**) is strictly decreasing in s. Remember we have already shown that the left hand side of (**??**) is increasing in V. So $V_m(s)$ must be increasing in s.

(iii) If $s_1, s_2 \in S_A$ or $s_1, s_2 \in S_I$, then $v(s_2) \geq v(s_1)$ follows directly right from (i) and (ii). Suppose $s_1 \in S_I$ but $s_2 \in S_A$. Then

$$v(s_2) = \rho V_m(s_2) + (1-\rho)V_n(s_2) \geq V_n(s_2) \geq V_n(s_1) = v(s_1).$$

Suppose $s_1 \in S_A$ but $s_2 \in S_I$. Suppose $v(s_1) > v(s_2)$. Then

$$\rho V_m(s_1) + (1-\rho)V_n(s_1) \geq V_n(s_2),$$

which in turn implies $V_m(s_1) \geq V_n(s_2)$. This contradicts $s_1 \in S_I$ since

$$U(V_m(s_1)) + s_2 - \beta\overline{U} > U(V_m(s_1)) + s_1 - \beta\overline{U} \geq 0.$$

Finally, since $\rho < 1$, v is strictly increasing on $\mathbb{R}_+$. This proves the lemma.

With the above lemmas, we now proceed to prove the proposition.

1. Let (Y, d) denote the space of all bounded and continuous functions $f : \mathbb{R}_+ \to X$ under the *sup* norm, denoted d. (Note that boundedness is needed for $\mathbb{R}_+$ is not compact.) Y is a complete normed vector space.

2. Define a mapping Γ as follows:

$$\forall v \in Y \text{ and } s \in \mathbb{R}_+, \, \Gamma(v)(s) = \begin{cases} \rho V_m(s) + (1-\rho)V_n(s) \, , \text{ if } s \in S_A \\ V_n(s) \qquad\qquad , \text{ if } s \in S_I \end{cases}$$

subject to (110)-(113).

Notice that given Lemma 6, the function $\Gamma(v)$ is well defined for all $v \in Y$.

3. We show that Γ maps from Y to Y, that is, $\Gamma : Y \to Y$. We must show that Γ preserves boundedness and continuity. That Γ preserves boundedness is obvious. We now show that Γ preserves continuity. Let $v \in Y$.

(3a) From Lemma 5, we know that the function $V_n(s)$ is continuous and strictly increasing on $\mathbb{R}_+$. We also know that the function $V_m(s)$ is continuous and increasing on S_A.

(3b) Observe that S_I is an open set in $\mathbb{R}_+$ and hence S_A is closed. To show this, let $s \in S_I$. Since $0 \in S_A$ by Lemma 4, we have $s > 0$. This implies $U(V) + s - \beta\overline{U} < 0$ for all $V \in \Phi$ such that $V \geq V_n(s)$. Given the continuity of U and V_n, there exists $\varepsilon > 0$ such that $(s-\varepsilon, s+\varepsilon) \subseteq S_I$.

(3c) Since $S_I \in \mathbb{R}_+$ is open, it can be written as an union of disjoint open intervals in $\mathbb{R}_+$.

(3d) Observe next that $[0, (V_n)^{-1}(V^*)] \subseteq S_A$. This is because : $\beta\overline{U} \leq U(V^*)$ by equation (22), the value function $U(V)$ is concave by Assumption 2, and the function $V_n(s)$ is continuous and strictly increasing by Lemma 5.

(3e) With (3c) and (3d), there exists a vector $\{b_0, a_i, b_i, \, i = 1, 2, ..., m\} \subseteq \mathbb{R}_+$ such that

$$S_A = [0, b_0] \cup \left(\bigcup_{i=1}^{m} [a_i, b_i] \right)$$

where

$$(V_n)^{-1}(V^*) \leq b_0 < a_1 \leq b_1 < \cdots < a_m \leq b_m$$

and the values of m and b_m may be infinity.

(3f) Clearly, $\Gamma(v)$ is continuous on $\mathbb{R}^+ \setminus \{b_0, a_1, b_1, \cdots, a_m, b_m\}$. So, $\Gamma(v)$ is continuous on $\mathbb{R}_+$ if and only if $V_m(s) = V_n(s)$ for $s \in \{b_0, a_1, b_1, \cdots, a_m, b_m\}$.

Suppose $V_m(b_i) > V_n(b_i) \geq V^*$ for $i \in \{0, 1, \cdots, m\}$, Since V_n is continuous, there exists $\varepsilon > 0$ such that $[b_i, b_i + \varepsilon) \subseteq S_A$, a contradiction.

Suppose $V_m(a_i) > V_n(a_i) \geq V^*$ for $i \in \{1, \cdots, m\}$. since $V_n(a_i) > V^*$, U is strictly decreasing for $V \geq V^*$. This implies that $U(V_n(a_i)) + a_i - \beta\overline{U} > 0$. There exists $\varepsilon > 0$ such that $(a_i - \varepsilon, a_i] \subseteq S_a$ which is a contradiction.

We have proved that the function $\Gamma(v)$ is continuous.

4. We show that the mapping Γ is a contraction. Since the underlying space is a normed vector space of bounded and continuous functions, we need only verify that the Blackwell sufficient conditions are satisfied.

(Monotonicity) Let $v_1, v_2 \in Y$ and $v_1 \geq v_2$. We must show that $\Gamma(v_1)(s) \geq \Gamma(v_2)(s)$ for all $s \in \mathbb{R}_+$.

Let S_A^i, S_B^i, V_n^i, V_m^i $(i = 1, 2)$ denote the sets S_A and S_B and the functions V_n and V_m induced by v_i through (110)-(113). Notice first that $V_n^1 \geq V_n^2$.

(i) Suppose $s \in S_A^1 \cap S_A^2$. Then the property of the CES objective function guarantees that $V_m^1(s) \geq V_m^2(s)$, and hence $\Gamma(v^1)(s) \geq \Gamma(v^2)(s)$.

(ii) Suppose $s \in S_I^1 \cap S_A^2$. We need only show $V_n^1(s) \geq V_m^2(s)$, which holds, for otherwise $s \in S_A^1$.

(iii) Suppose $s \in S_I^1 \cap S_I^2$. In this case $\Gamma(v^1)(s) = V_n^1(s) \geq V_n^2(s) = \Gamma(v^2)(s)$.

(iv) Suppose $s \in S_A^1 \cap S_I^2$. In this case, $V_m^1(s) \geq V_n^1(s) \geq V_n^2(s)$, implying $s \in S_A^2$, a contradiction. So $s \in S_A^1 \cap S_I^2$ cannot be the case.

We therefore have shown that the mapping Γ is monotonic.

(Discounting) Let $v_1, v_2 \in Y$ and let $v_2 = v_1 + a$ for any $a > 0$. We show that $\Gamma(v^2)(s) \leq \Gamma(v^1)(s) + \beta\Delta a$ for all $s \in \mathbb{R}_+$.

Observe first that $V_n^2(s) = V_n^1(s) + \beta\Delta a$ for all $s \in \mathbb{R}_+$.

Consider first the case $s \in S_A^1 \cap S_A^2$. The desired result in the case holds trivially if the maximized Nash product is zero. In the following, we consider the case where the maximized Nash product is strictly positive.

Let

$$\varphi_i = -\frac{1-\omega}{\omega} \frac{U(V_m^i(s)) + s - \beta\overline{U}}{V_m^i(s) - V_n^i(s)}, \ i = 1, 2,$$

where φ_i for $i = 1, 2$ is the slope of the value function $U(V)$ at optimum, i.e., at $V = V_m^i(s)$.

Given the concavity of U and the differentiability of indifference curve, U has to be under the following straight lines

$$f_i(x) = \varphi_i(x - V_m^i(s)) + U(V_m^i(s)), \ i = 1, 2$$

Suppose $V_m^2(s) > V_m^1(s) + \beta\Delta a$. Then $\varphi_1 < \varphi_2 < 0$. Therefore,

$$U(V_m^2(s)) \leq f_1((V_m^2(s))) \Rightarrow U(V_m^1(s)) > f_2((V_m^1(s))),$$

a contradiction. So, we conclude that $V_m^2(s) \leq V_m^1(s) + \beta\Delta a$.

The cases $s \in S_A^1 - S_A^2$, $s \in S_A^2 - S_A^1$, and the case $s \notin S_A^1 \cup S_A^2$ are straightforward to analyze and are left for the reader. This proves that the mapping Γ has the discounting property and hence we have shown that Γ is a contraction.

5. By the contraction mapping theorem then, $v \in Y$ and is the unique fixed point of Γ. So v is continuous and the proposition is proved.

2.7.3 Proof of Proposition 2.3

Suppose $i \in \Omega$ but $U(V_i) < \overline{U} - v^{-1}(V_i)$, then move i from Ω to Ω' while not changing the values of a, c_i and V_i. The modified contract would remain feasible, but the firm's value is strictly increased. On the other hand, suppose $U(V_i) > \overline{U} - v^{-1}(V_i)$ but $i \notin \Omega$. Then move i from Ω' to Ω to increase the firm's value. The proposition is proven.

2.7.4 Proof of Proposition 2.4

That the worker is terminated with expected utility V implies

$$U(V) < \overline{U} - v^{-1}(V).$$

Now this worker would go immediately to the market to look for a new match if and only if

$$\exists\, V' \geq V_n(v^{-1}(V)) \text{ such that } U(V') + v^{-1}(V) \geq \beta\overline{U}.$$

But this implies the existence of $V_m(v^{-1}(V))$ and it holds that $V_m(v^{-1}(V)) \geq V$. Let $V' = V_m(v^{-1}(V))$. Then

$$U(V') + v^{-1}(V) \geq \beta\overline{U}.$$

Suppose $\overline{U} = 0$. Hence, $U(V') > U(V)$. The proposition is proven.

2.7.5 Proof of Proposition 2.5

We first show that with the equilibrium contract where $\overline{U} = 0$, it holds that

$$\widehat{\Omega} \subseteq S_A.$$

This is easy to show. For each $v^{-1}(V) \in \widehat{\Omega}$, we have (i) $V \in [v(0), V_{max}) \subseteq \Phi$; (ii) $V = v(v^{-1}(V)) \geq V_n(v^{-1}(V))$; and (iii) $U(V) + v^{-1}(V) \geq 0$. So $v^{-1}(V) \in S_A$. Remark: The fact that $\overline{U} = 0$ is important for part (iii) of the proof.

Next, we show

$$\widehat{\Omega}_A \subseteq S_A.$$

To show this, let $s = 0$ and $V = V_* \in \Phi$. Clearly, $V_* \geq V_n(0)$ and $U(V_*) + 0 \geq \beta\overline{U}$ by (23). Remark: the equilibrium condition $\overline{U} = 0$ is not needed in the proof here.

Last, with the equilibrium contract with $\overline{U} = 0$, it holds that

$$\widehat{\Omega}_I \subseteq S_I.$$

This is a corollary of Proposition 2.4 which says that if $s \in \widehat{\Omega}$ then $s \notin S_A$ and hence $s \in S_I$. The proposition is proven.

2.7.6 Proof of Proposition 2.6

We prove the first part of the proposition by way of contradiction. Suppose otherwise. Then there exists a strictly monotonic sequence $\{s_q\}$ such that $s_q \in S_I$ for all q, and $s_q \to 0$ as $q \to \infty$.

Next we show that it must hold that $V_n(s_q) \to 0$ as $q \to \infty$. Since $s_q \in S_I$ $\forall q$,

$$v(s_q) = V_n(s_q) = \max_{0 \le c \le s_q} \{u(c) - \phi(0) + \beta\Delta v[(1+r)(s_q - c)/\Delta]\}.$$

Let $q \to 0$ on both sides of the above equation to obtain

$$u(0) - \phi(0) + \beta\Delta v(0) = v(0)$$

or

$$v(0) = [u(0) - \phi(0)]/(1 + \beta\Delta) = 0.$$

So $V_n(s_q) \to v(0) = 0$ as $q \to \infty$.

Now for each q and s_q, consider the following contract for the unemployed worker with assets s_q. The worker is employed for one period. For the period the worker is employed, his compensation is determined by σ_0. The worker is then terminated with assets s. So the worker's utility under this contract is $H_0 + \beta\Delta v(s_q)$. For q large enough, it must then hold that

$$H_0 + \beta\Delta v(s_q) \ge (1 - \beta\Delta)V_n(s_q) + \beta\Delta V_n(s_q) = V_n(s_q).$$

This holds because (a) $s_q \in S_I$ so $v(s_q) = V_n(s_q)$; (b) $H_0 > 0$; and (c) $V_n(s_q) \to 0$ as $q \to \infty$.

Finally, notice that for q large enough, it holds that

$$\Pi(H_0) + s_q - \beta\Delta s_q + \beta\overline{U} > \beta\overline{U}.$$

Thus we have shown $s_q \in S_A$ for q large enough. A contradiction. This proves the first part of the proposition.

We now prove the second part of the proposition. We know that $0 \in S_A$. Let $s_* \equiv \min\{s \in S_I\}$. Given that $v(\cdot)$ is continuous, it must hold that

$$V_n(s_*) = \rho V_m(s_*) + (1 - \rho V_n(s_*),$$

or

$$V_n(s_*) = V_m(s_*).$$

It then must hold that

$$U(V_n(s_*)) + s_* = \beta \overline{U} = 0..$$

Let $f(s) \equiv U(V_n(0)) + s$ for all $s \geq 0$. Observe first that

$$f(0) = U(V_n(0)) + 0 = U(V^*) > 0,$$

where $V^* > V_n(0)$. Observe next that over the interval $[0, v^{-1}(V^*)]$, $f(s)$ is strictly monotone increasing in s, for $V_n(s)$ is strictly increasing in s at all s and $U(V)$ is strictly increasing in V over the interval $[V_n(0), V^*]$. Therefore a solution (**??**) cannot exist over the interval $[0, v^{-1}(V^*)]$. That is

$$s_* > v^{-1}(V^*).$$

This proves the proposition.

2.7.7 Proof of Proposition 2.7

We need only show that there exists $\overline{V} > v(0)$ such that

$$U(V) \leq \overline{U} - v^{-1}(V), \ \forall V > \overline{V}.$$

In turn, we need only show that there exist functions $\hat{U}(\cdot)$ and $C(\cdot)$ such that for V sufficiently large, it holds that

$$U(V) \leq \hat{U}(V) \leq \overline{U} - C(V) \leq \overline{U} - v^{-1}(V).$$

We now construct the functions $\hat{U}(\cdot)$ and $C(\cdot)$. First, set for each $V \geq v(0)$,

$$C(V) \equiv \frac{u^{-1}[(1-\beta\Delta)V]}{1-\beta\Delta}.$$

$C(V)$ is the cost to the firm of terminating a worker who is not to participate in the labor market in the rest of his life, so it must hold that $C(V) \geq v^{-1}(V)$ for all V.

Next, for each $V \geq v(0)$, define $\hat{U}(V)$ as follows. Imagine a firm who currently employs a worker with expected utility V at the beginning of a period. Suppose this worker is not subject to moral hazard. That is, when this worker is employed by the firm, his efforts are perfectly observable to the firm so the the firm can make the worker's compensation contingent on the worker's efforts. With his worker, the optimal termination strategy for the firm is to either never terminate the worker until he dies, or terminate him after one period.

Consider first the case where the worker is never terminated. In this case, the worker is given a constant compensation say c and he is asked to make a constant effort say a, where c and a satisfy

$$V = \frac{u(c) - \phi(a)}{1 - \beta\Delta}.$$

Since $a \geq \underline{a}$,

$$c \geq u^{-1}((1 - \beta\Delta)V + \phi(\underline{a})).$$

In this case then the value of the firm is less than or equal to

$$\hat{U}_1(V) \equiv \frac{\overline{\theta} - u^{-1}((1 - \beta\Delta)V + \phi(\underline{a}))}{1 - \beta\Delta}.$$

Consider next the case in which the worker is terminated after one period. Consider the best possible scenario for the firm where, after termination, the worker is employed every period by some other firm which gives the worker all the surplus of the match so the worker's current employer can incur the least possible cost of termination. Suppose this translates into a constant compensation of $\overline{c}$ with a constant level of effort $a_* \geq \underline{a} > 0$ for the worker after termination. Now let a_1 denote optimal effort in the first period. Let c_1 denote the optimal compensation for the worker in the first period, and c_2 the optimal compensation in each period after the termination. Then

$$V = u(c_1) - \phi(a_1) + \frac{\beta\Delta}{1 - \beta\Delta}[u(c_2 + \overline{c}) - \phi(a_*)].$$

To minimize cost, the firm sets $c_1 = c_2 + \overline{c}$, and hence

$$V + \phi(a_1) + \frac{\beta\Delta}{1 - \beta\Delta}\phi(a_*) = \frac{u(c_2 + \overline{c})}{1 - \beta\Delta};$$

and so

$$c_2 = u^{-1}[(1 - \beta\Delta)(V + \phi(a_1)) + \beta\Delta\phi(a_*)] - \overline{c}.$$

The value of the firm in this case is therefore equal to

$$\begin{aligned}
&\overline{\theta} - (c_2 + \overline{c}) - \frac{\beta\Delta}{1 - \beta\Delta}c_2 \\
=&\overline{\theta} - \overline{c} - \frac{u^{-1}[(1 - \beta\Delta)(V + \phi(a_1)) + \beta\Delta\phi(a_*)] - \overline{c}}{1 - \beta\Delta} \\
\leq&\overline{\theta} - \frac{\beta\Delta}{1 - \beta\Delta}\overline{c} - \frac{u^{-1}[(1 - \beta\Delta)V + \phi(\underline{a})]}{1 - \beta\Delta} \\
\equiv&\hat{U}_2(V) \qquad (117)
\end{aligned}$$

Let $\hat{U}(V) = \max\{\hat{U}_1(V), \hat{U}_2(V)\}$ for all V. It is easy to see that $\hat{U}_1(V) > \hat{U}_2(V)$, so $\hat{U}(V) = \hat{U}_1(V)$. Therefore equation (**??**) holds if and only if

$$\frac{\overline{\theta} - u^{-1}((1-\beta\Delta)V + \phi(\underline{a}))}{1-\beta\Delta} \leq \overline{U} - \frac{u^{-1}((1-\beta\Delta)V)}{1-\beta\Delta}.$$

or

$$\overline{\theta} - (1-\beta\Delta)\overline{U} \leq u^{-1}[(1-\beta\Delta)V + \phi(\underline{a})] - u^{-1}[(1-\beta\Delta)V].$$

This proves the proposition.

2.7.8 Proof of Proposition 2.8

Let $\overline{s} = \max_{s \in S_A} s$, which is well defined since S_A is compact. Consider the optimization problem faced by a non-employed worker with asset $s > \overline{s}$,

$$\max_{c \geq 0, t \in \{1,2,\cdots\} \cup \{\infty\}, s'} \left(\frac{1-k^t}{1-k}[u(c) - \phi(0)] + k^t v(s') \right)$$

subject to

$$\frac{1-k^t}{1-k}c + k^t s' = s$$

$$s' \in S_A$$

where $k = \beta\Delta = \frac{\Delta}{1+r} < 1$.

Note that if $t = \infty$, then $k^t = 0$, and the choice of s' is not relevant. Note that it is optimal to make consumption constant across the periods before reentering the labor market with asset $s' \in S_A$ at the period $t+1$. The case $t = \infty$ is the case where the worker consumes the annuity of his/her asset every period and never comes back to the labor market.

For the purpose of this proof, we can replace the constraint $t \in \{1, 2, \cdots\} \cup \{\infty\}$ in the optimization problem above by $t \in \mathbb{R}_+ \cup \{\infty\}$. Observe next that it is optimal to have $s' = \overline{s}$, by the definition of the function $v(\cdot)$.

Let the first derivative of the objective function with respect to t be denoted $H(s,t)$,

$$\begin{aligned} H(s,t) &\equiv -k^t \ln k \left(\frac{u[\frac{1-k}{1-k^t}(s - k^t\overline{s})] - \phi(0)}{1-k} - v(\overline{s}) - \frac{s - \overline{s}}{1-k^t} u'[\frac{1-k}{1-k^t}(s - k^t\overline{s})] \right) \\ &\equiv -k^t \ln k F(s,t). \end{aligned}$$

Notice first that $-k^t \ln k > 0$ for all $t \in \mathbb{R}_+$. Notice next that $F(s,t)$ a strictly decreasing in t since its partial derivative with respect to t is

$$-\frac{(1-k)k^t \ln k (s-\overline{s})^2}{(1-k^t)^3} u'' \left(\frac{1-k}{1-k^t}(s - k^t\overline{s}) \right) < 0$$

and goes to

$$L(s, \overline{s}) = \frac{u[(1-k)s] - \phi(0)}{1-k} - v(\overline{s}) - (s - \overline{s})u'[(1-k)s]$$

as t goes to infinity. Last, observe that

$$F(s, 0) = \frac{u(\infty) - \phi(0)}{1-k} - v(\overline{s}) > 0.$$

The above observations imply that $t = \infty$ is optimal if and only if $L(s, \overline{s}) \geq 0$.

So consider the function L. Observe first that

$$L(\overline{s}, \overline{s}) = \frac{u[(1-k)\overline{s}] - \phi(0)}{1-k} - v(\overline{s}) \leq 0,$$

where the inequality follows from the definition of the function $v(\cdot)$. Next, notice

$$L_1(s, \overline{s}) = -(1-k)(s - \overline{s})u''[(1-k)s] > 0.$$

Third, notice

$$\lim_{s \to \infty} L(s, \overline{s}) = V_{\max} - v(\overline{s}) > 0$$

which holds because $\lim_{s \to \infty} (s - \overline{s})u'[(1-k)s] = 0$ given $u(\cdot)$ is bounded.

With the above observations, we conclude that $L(s) \geq 0$ if and only if $s \geq s^*$ where s^* solves

$$L(s^*, \overline{s}) = 0,$$

and $s^* > \overline{s}$. The proposition is proved.

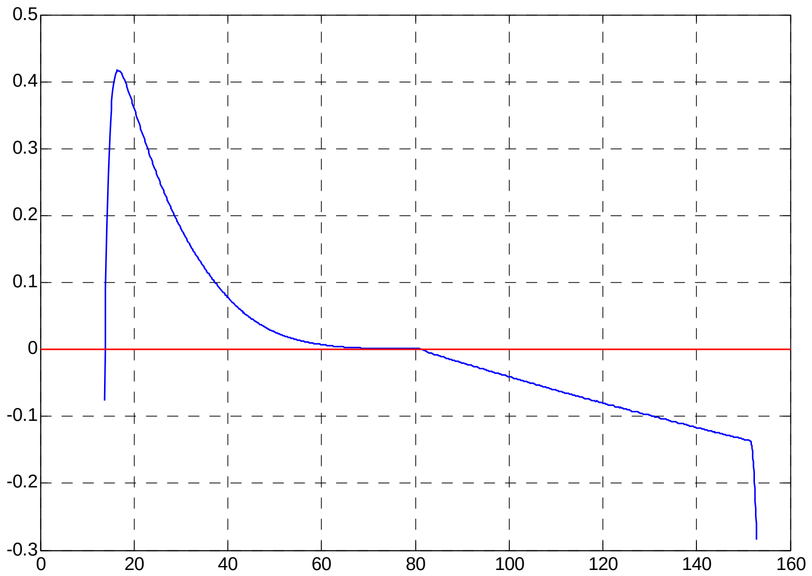
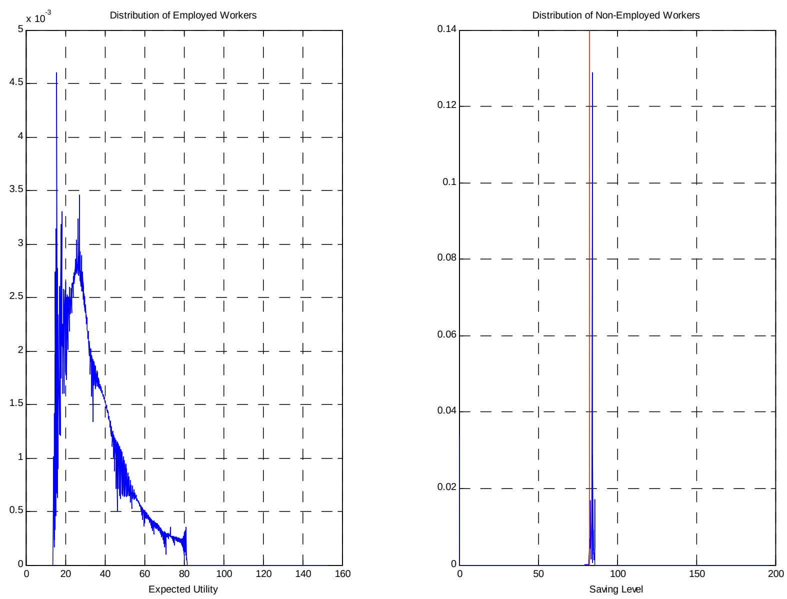


Figure 2.1



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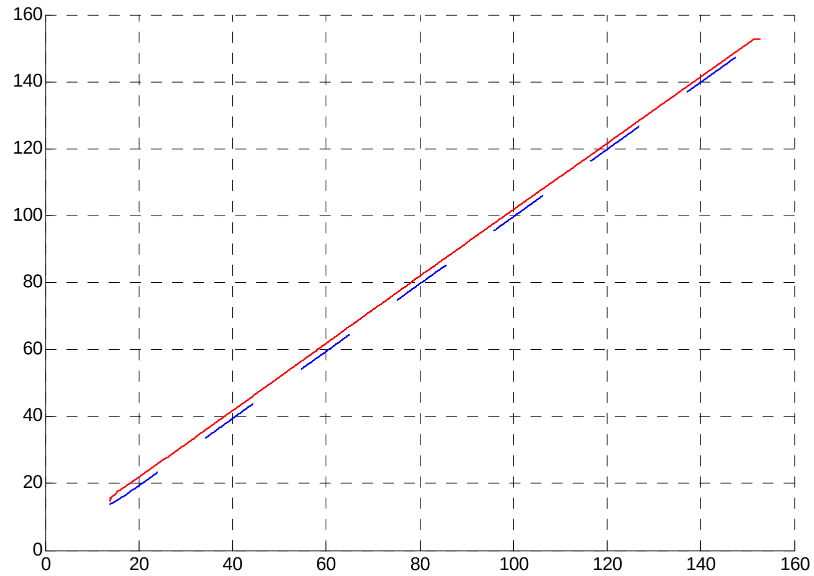

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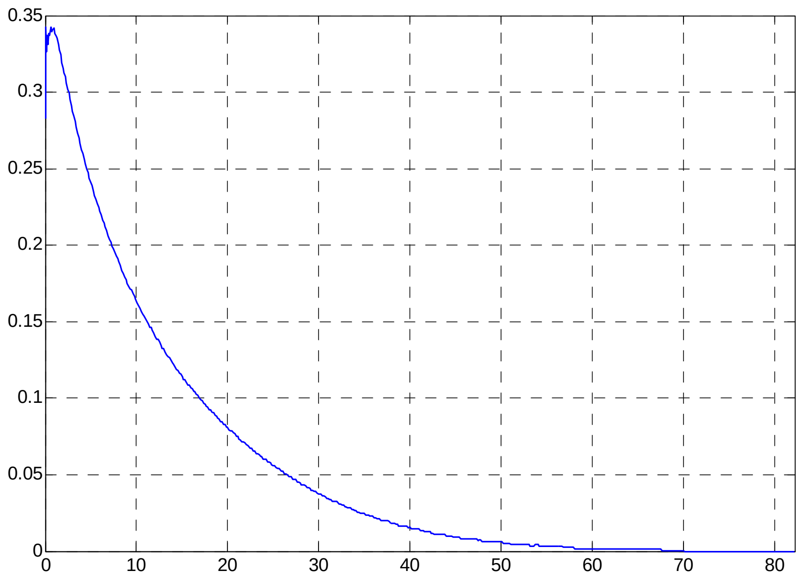

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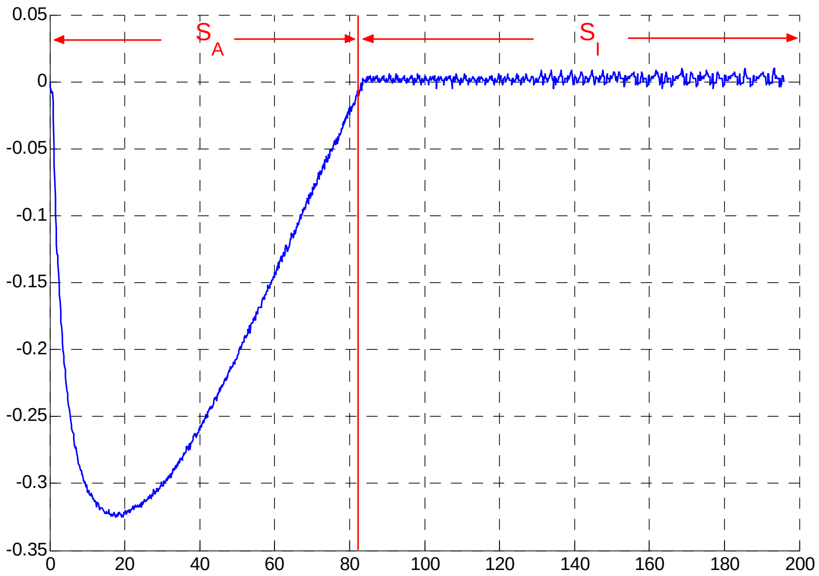

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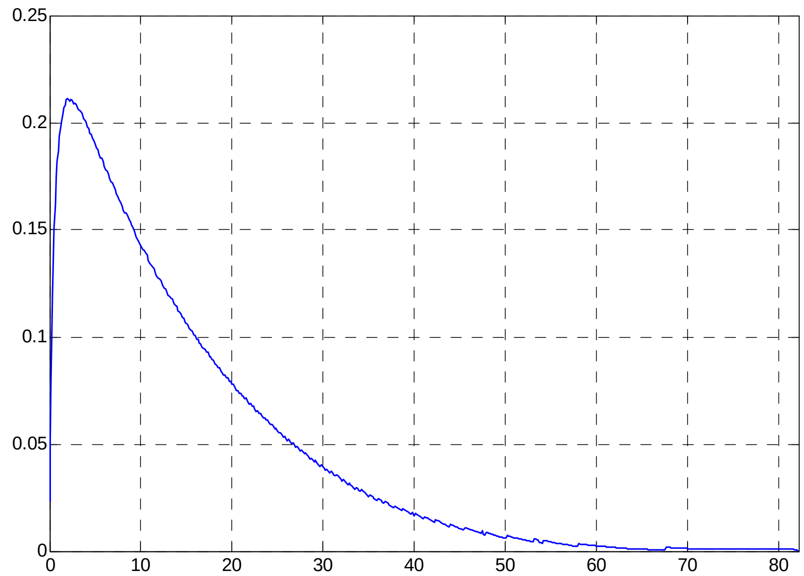

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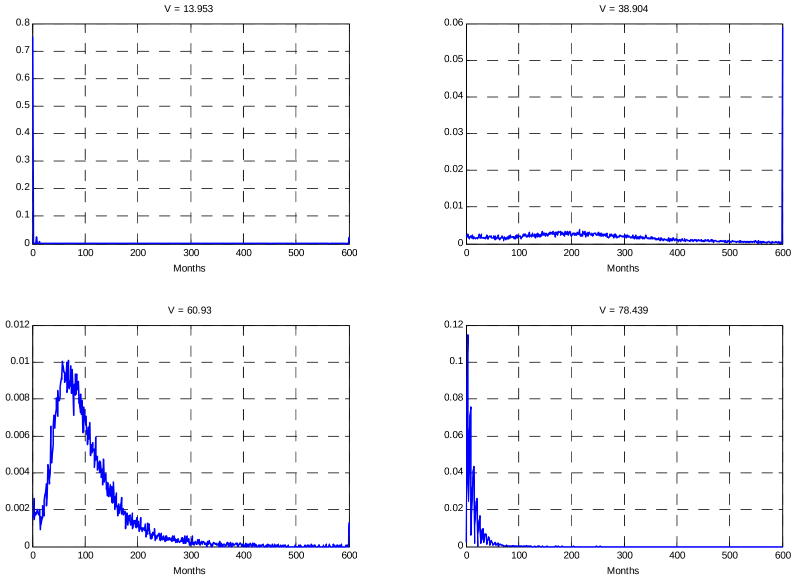

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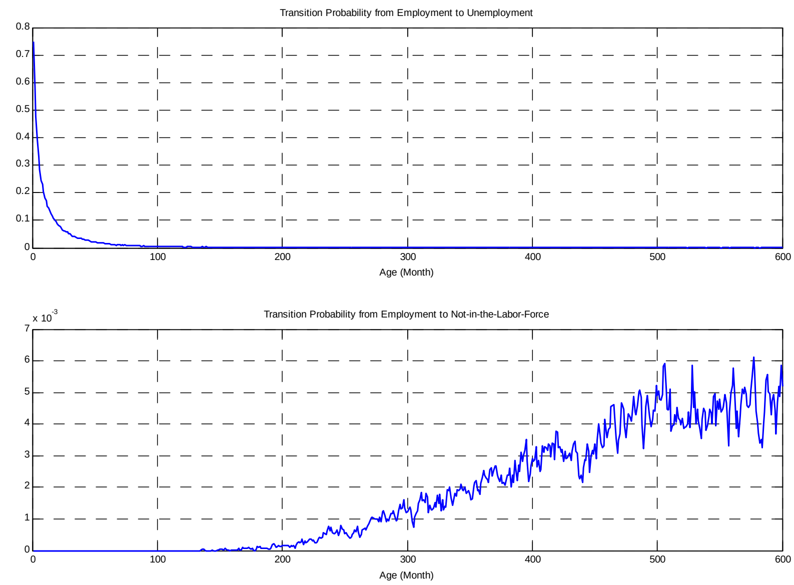

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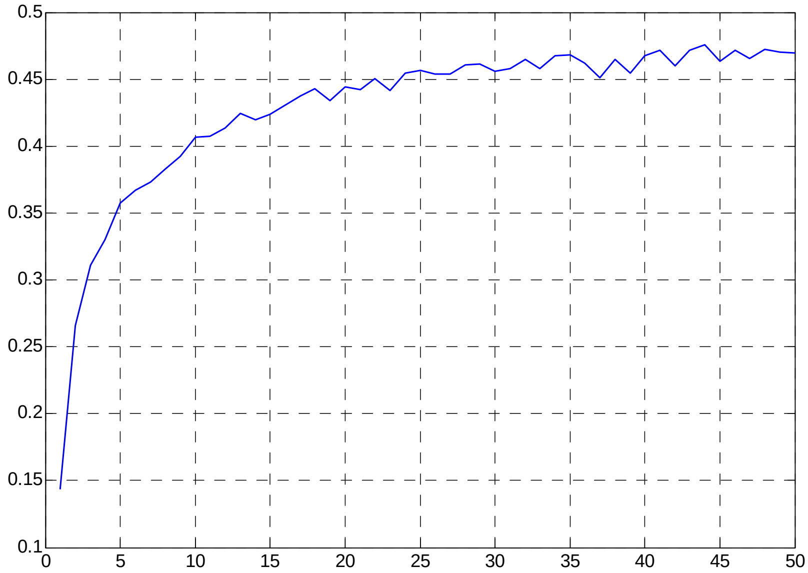

Figure 2.9

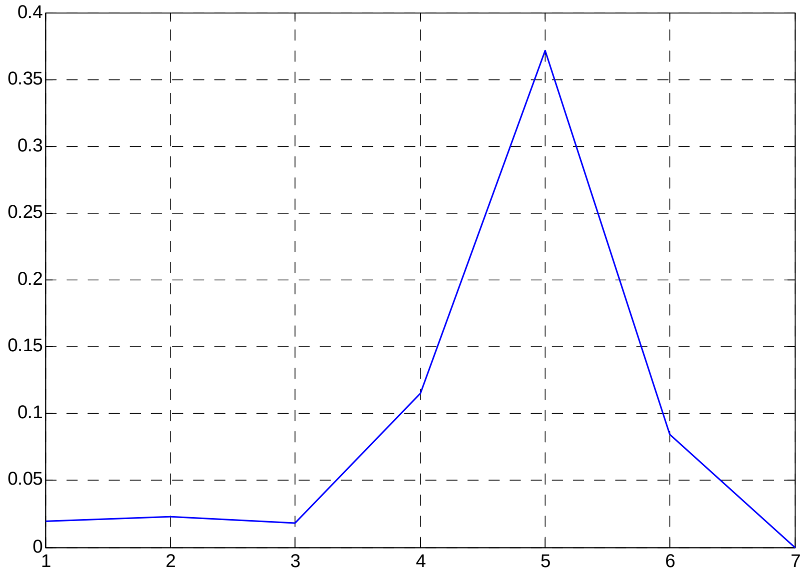

Figure 2.10

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